Sandpile models on fractal lattices
A case study: the Sierpinski gasket
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Preface

This thesis on sandpile models for self-organised criticality concludes my research at the Limburgs Universitair Centrum in Diepenbeek, Belgium. Besides sandpiles I also studied other models for self-organised criticality such as the Drossel and Schwabl forest fire model [1], the Bak-Sneppen model [2, 3] and the Earthquake model of Olami, Feder and Christensen [4]. The research on sandpile models on fractal lattices however formed the major part and it forms a unity on its own, and therefore this thesis concentrates on this. In particular I present analytical and numerical results for the Bak-Tang-Wiesenfeld (BTW) model, and numerical results for the Manna model.

The thesis essentially consists of three parts. The first part, covering chapters 1, 2 and 3, gives a review of existing sandpile theory. This is not intended as merely an introduction, as I spent an important part of my research understanding the existing work. Hereby I concentrated on Dhar's group-theoretical results on the BTW model, the theory of waves in this model, and the problems of scaling in the BTW and Manna models. An understanding of this research requires knowledge of various topics in physics and mathematics, such as graph theory, resistor networks, discrete physics, multifractal scaling, ... Therefore I thought it was worthwhile to collect them in one place together with the sandpile theory. Chapter 1 is an introductory chapter, chapter 2 presents the auxiliary tools, and chapter 3 resumes important general results in sandpile description. Moreover there was some adaptation and extension of the theory in order to be applicable to a general fractal lattice. This was necessary since most papers handle sandpile models on the two dimensional square lattice and hence don’t yield general expressions for e.g. various exponents.

The main new results of this thesis from a general theoretical point of view lie in the description of waves of topplings on a general fractal lattice (see sections 3.2 and 3.3). I will present new results for statistical properties of waves, as well as last waves, and successive waves correlations.

The next part of the thesis, chapter 4, is a case study in which I use the
general theory discussed so far to derive various stationary state properties of the BTW model on a specific fractal lattice, namely the Sierpinski gasket. Together with this the results of extensive numerical simulations are presented. Using them I derived properties which so far cannot be computed analytically. Various analysis methods are presented which are very convenient for analysing distribution functions in models on fractal lattices. On such lattices the distribution functions behave quite irregularly and are hard to analyse by e.g. simple collapse plots or curve fits. Most important here is a thorough analysis of the scaling behaviour in the BTW model. This part ends with a presentation of the results of extensive simulations of the Manna sandpile model. Also here the scaling behaviour is thoroughly analysed.

The last part of this thesis is chapter 5 in which a new analysis of the BTW model with dissipation is presented. This analysis uses a mapping of the model on a random walk problem. A general expression for the crossover exponent between the dissipative and nondissipative model is derived, which differs from an earlier conjecture. Using again the case study of the Sierpinski gasket, it is found that our expression is correct on this lattice, and by means of a real space renormalisation calculation we show explicitly that the earlier conjecture does not hold in this case.

Parts of my research have been published in four papers, namely


Also parts have been presented at several meetings, e.g. the general scientific meeting of the Belgian Physical Society (Brussels, 1999), the spring colloquium on statistical mechanics of the K. U. Leuven (1999) and the research workshop on self-organised criticality and phase transitions in driven systems at ICPT (Trieste, 2000).
I would like to thank my promotor Carlo Vanderzande in special here. He guided me with patience through these modern problems of nonequilibrium statistical mechanics. I also would like to thank the other people of the theoretical physics group at the L.U.C., including Marc Bouten, Roger Serneels, Chris Vandenbroeck, Geert-Jan Bex, Jan Schietse, Eddy Lootens, Peter Reimann, Anatoly Patrick, Bart Van Rompaey, Mauro Copelli, Peter Leoni, Jef Hooyberghs, Ioana Birzu and Bart Cleuren. I would like to thank Geert-Jan, Peter L. and Roger in special for the help concerning computers. Other people I enjoyed collaborating with at the L.U.C. include Klaus Schmidt, Gilbert Knuyt, Marc Gyssens, Herman Janssen and the helpful people of the W.N.I. secretariat.

Regarding scientific work I would like to thank besides Carlo also among others Vyacheslav Priezzhev, Deepak Dhar, Sven Lübeck, Dimitri Ktitarev, Attilio Stella, Mario De Menech and David Rail for interesting and useful discussion. In special I would like to thank Vyacheslav for collaboration.

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I would like to thank Dominique Fonteyn of the Belgian Institute for Space Aeronomy for aid on the required supplementary proposition to this thesis. This supplementary proposition reads as follows: "Chemical data assimilation of MLS O_3, HNO_3 and ClO observations in a nonlinear chemical regime, due to chlorine activation by polar stratospheric clouds, is possible".

Frank Daerden
Brussels, september 2001
Chapter 1

Sandpiles and self-organised criticality

1.1 Introduction

Next to its classical theoretical and experimental fields a new way of physics emerged with the introduction of computers. The possibility of performing complex and large-scale computations arose. The physicist had a tool to analyse complex systems and nonlinear phenomena.

By example now there was the possibility of appropriately modeling the atmosphere and letting the models evolve very fast in the computer in order to make weather forecasts.

This third way of working - sometimes referred to as numerical experimenting - brought physics close to other sciences, because many sciences deal with complex systems [5]. Think of e.g. exact science disciplines as geography, meteorology, biology, Earth system science and ecology, …, but also of social sciences, economics, history, ….. The link to biology is highly interesting since biological systems seem to manifest the highest forms of complexity we know. This whole new cross-disciplinary area of research is now commonly denoted by the science of complexity.

Adequately modeling a system and simulating it in a computer can provide us with nice information. However the physicist is still interested in analytical results. To gain more insight and understanding in the very basic physics that is going on. What are the basic characteristics of a system? What can we expect when we change parameters or circumstances? What can we predict without computing? This is one of the aims of the idea of self-organised criticality.
| Planetary dynamics:   | Earthquake dynamics [4]  
|                     | Volcanic activity  
|                     | River networks  
|                     | Mountain ranges structure  
|                     | Meteorite size distributions [6]  
|                     | Impact crater size distribution [7]  
|                     | Structure of tropospheric clouds [8]  
|                     | Forest fire structure and evolution [1]  
|                     | ...  
| Dynamics of life:   | Evolutionary dynamics [2, 9, 3]  
|                     | evolution and extinction of species  
|                     | biodiversity and ecosystem structure  
|                     | Structure of organic tissues [10]  
|                     | Brain activity (brain waves) [11]  
|                     | Structure of languages, books, ...  
|                     | Structure of music, paintings [12], ...  
|                     | Economy - econophysics  
|                     | ...  
| Stellar dynamics:   | Solar flare dynamics [13]  
|                     | Structure of interstellar matter [8]  
|                     | Star formation (initial mass function)  
|                     | Pulsar glitches  
|                     | Light of quasars  
|                     | Large-scale structure of the universe [14]  
|                     | ...  

Table 1.1: Some examples of physical systems where critical properties are observed or supposed. Further references on unreferenced examples can be found in e.g. [10, 15, 16].
1.2 Self-organised criticality

The theory of self-organised criticality (SOC) is concerned with a large class of complex systems that are described by simple power laws. The main characteristics of the behaviour of these systems are:

- self-similar or fractal spatial behaviour,
- self-similar temporal behaviour resulting in 1/f noise,
- unpredictability and intermittent behaviour.

So the complex behaviour in these systems is rather nicely structured. Indeed, self-similar behaviour can be described by simple power laws. We call this kind of complex behaviour critical because of its resemblence with the behaviour of a system in thermodynamical equilibrium at a second order phase transition, i.e. at a critical point. The abundance, the nature and importance of systems where this kind of complexity is found or supposed are so impressive that we have to regard critical behaviour as an important example of complexity. A number of examples are listed in table 1.1.

The story of SOC starts with the analogy between the behaviour observed in systems at an equilibrium second order phase transition and the kind of behaviour observed in complex systems like the ones listed in table 1.1. In a system in equilibrium a control parameter has to be fine-tuned to a specific value in order to make the system critical. Taking into account the analogy in behaviour, it looks as though the dynamics of the complex systems of table 1.1 drives them to a critical point. Therefore the name self-organised criticality was chosen [15, 17]. In order to understand this phenomenon a mechanism was proposed where the control parameter is regulated by a feedback mechanism via the order parameter, in such a way that the critical point is the attractor of the dynamics [18].

In 1987 Per Bak, Chao Tang and Kurt Wiesenfeld proposed a simple model in which the dynamics leads the system towards a stationary state in which the system manifests critical properties [17]. Since the model was inspired by the dynamics of a pile of sand to which sandgrains are added, it was called the sandpile model, now known as BTW (sandpile) model. This paper ignited a real avalanche of proposals for other models for SOC. There were variations on the original sandpile model, with important parameters changed [19, 20], but also simplified models for real systems like some in table 1.1 were proposed, e.g. for Earthquake dynamics [4] and the evolution of life [2, 9, 3].

In the past 13 years criticism arised as to that the BTW sandpile model was not good as a prototype model of SOC, mainly for the simple reason
that it did not totally expose the properties SOC is intended to describe (e.g. finite-size scaling). This point forms one of the subjects of this thesis. However the BTW model is still the most analytical tractable model of all proposed models for SOC. In fact it has more or less also started a life on its own, outside of SOC, because it shows a lot of interesting behaviour and combines nice topics in physics, graph theory etc.

Recently it was proposed that the Manna (sandpile) model would be a better candidate as prototype model for SOC. It shows very nicely typical critical features. This again is one of the points that will be investigated in this thesis.

1.3 General ideas behind sandpile models

Basically all sandpile models are made up according to the same idea. There is an input of energy (grains of sand) into a system of many elements. The energy storage capacity of these elements is limited and the system has to dissipate energy when it becomes "locally overheated". Generally there is a threshold mechanism that represents this overheating, this mechanism ensures that when the energy input in such an element exceeds its maximum storage capacity, it passes its energy towards its environment.

Very important is the system’s boundary, since the surplus of energy can only leave the system through this boundary. Sites near the boundary will be able to dissipate the energy towards the boundary quite easy: only the cooperation of a few sites is required. However for sites far away from the boundary, a major effort of a large part of the system will be necessary to dissipate the energy. The individual elements will have to cooperate. This cooperation must be installed from itself, not controlled by anyone or anything, since the individual elements only interact very locally. In other words, a transition will have to occur from a local interaction to a global cooperation (holism).

In many sandpile models this transition is observed. Without intervention from outside the system the individual elements manage to cooperate. The system organises itself into a state of global cooperation with critical properties. With critical we mean properties that are very similar to that of a system in equilibrium at its critical point, i.e. at a second order phase transition. Characteristics are algebraic, long-range correlations in space and time.

One can ask what the important ingredients are in a sandpile model in order to come to such a state of global cooperation. We can talk here of
1.4. DIFFERENT SANDPILE MODELS

'invisible' ingredients, since many sandpile models look quite general but in fact are based on not always obvious assumptions. One of these is a distinct time-scale separation. The dissipation of energy is assumed to occur on a much faster scale than the accumulation of energy. When this time-scale separation is not present the self-organised critical character of the models seems to disappear [21].

1.4 Different sandpile models

Various types of sandpile models have been introduced in the literature. In this thesis we focus on two types, namely the BTW model and the Manna model. As will be clear from their definitions, the main difference between these two models lies in the character of the sand redistribution rule. For the BTW model this is deterministic while in the Manna model it is stochastic. Common for these models is that the energy transport occurs in a discrete way.

1.4.1 The BTW sandpile model

The BTW sandpile model can be defined on any graph. The sites or vertices $i$ of the graph are occupied by a number of sand grains $z_i \in \mathbb{N}$. To each vertex a critical value $z_{ci}$ is appointed. One new grain of sand is added to an arbitrary site $i$. If the total number of sand grains at that site exceeds the critical value of that site (threshold dynamics!), it has to dissipate energy. In this case the site $i$ is said to be unstable. The dissipation occurs by a toppling mechanism: $-\Delta_{ij}$ grains of sand are transmitted from site $i$ to site $j$. It is important to note that in the BTW model the numbers $\Delta_{ij}$ are fixed and always used when site $i$ topples. Hence the redistribution rule is deterministic.

It is in this redistribution rule that the main difference lies between different BTW-inspired sandpile models. If the redistribution occurs in one or some directions, we have a directed sandpile model [22]. Also a random neighbour version can be defined, in which the sites where grains are redistributed to are randomly chosen over the whole system.

The most studied case is the uniform one where an equal number of grains is given to all nearest neighbours $j$ of site $i$. Mostly $\Delta_{ij}$ is then just chosen to be $-1$ for $i$ and $j$ nearest neighbours, and zero otherwise. Then, $z_{ci}$ is chosen to be the number of nearest neighbours of the site $i$. It is common to take $z_{ci}$ equal for all sites of the graph, and equal to the coordination number of the lattice. The sites with a lower number of nearest neighbours
Figure 1.1: Graphs with additional sink site. a: General graph with sink site indicated by \(\odot\), the coordination number in the graph is 3. b: Square lattice, the sink site is represented by the thick outer box. In both cases white circles are boundary vertices and edges from boundary vertices to the sink site are indicated by arrowed lines.
hence dissipate energy, they are said to belong to the system’s boundary. Sometimes one introduces a sink site connected by one or more bonds to sites in the system’s boundary, such that all sites in the system have the same coordination number. This sink site is not a real part of the system, e.g. it is not allowed to topple.

Eventually some or all of the neighbouring sites of \( i \) become unstable. Then each of them has to topple too. And so on, … eventually establishing a large avalanche of topplings.

The avalanche will be over when all the sites are stable again. Then one drops one new grain of sand on a arbitrary site, and the avalanche process is repeated. With this an explicit time scale separation is introduced in the model: the avalanche takes place on a fast timescale whereas the accumulation of energy occurs on a slow timescale. This is what is observed in many systems in Nature that seem to establish criticality through avalanche dynamics (see section 1.5).

Originally it was concluded from numerical simulations that in the BTW model the system self-organises into a critical state characterised by power law distributions and correlations, the SOC state.

Let us denote the sink site as 0, and the other sites by \( j, j = 1, \ldots, N \). The important parameters of the BTW sandpile model can be arranged in a matrix \( \Delta \), that is defined as follows (for the uniform case with \( z \) the coordination number in the lattice) \(^1\):

\[
\Delta_{ij} = \begin{cases} 
  z & \text{if } i=j \\
  -1 & \text{if } i \text{ and } j \text{ are nearest neighbours} \\
  0 & \text{otherwise}
\end{cases}
\]

(1.1)

and, when the site \( i \) topples, the toppling rule is given by:

\[
z_j \rightarrow z_j - \Delta_{ij}, \quad i > 0
\]

(1.2)

As already mentioned 0 is not allowed to topple. \( \Delta \) has the property that

\[
\sum_{j=1}^{N} \Delta_{ij} \geq 0 \quad \forall i.
\]

\(^1\) In fact the matrix \( \Delta \) can be regarded as the discrete version of the Laplacian operator. In two dimensions for small \( \epsilon \), \( \Delta f \approx (f(x+2\epsilon,y)+f(x,y+2\epsilon)+f(x-2\epsilon,y)+f(x,y-2\epsilon)-4f(x,y))/4\epsilon^2 \). Then for \( \epsilon = 1/2 \) the Laplacian operator acting on \( f \) in the point \((x,y)\) can be calculated by \( \Delta f \approx f(x+1,y)+f(x,y+1)+f(x-1,y)+f(x,y-1)-4f(x,y) \), i.e. by adding the value of \( f \) in the points one unit away from \((x,y)\) in each axis direction and subtracting four times the value of \( f \) in \((x,y)\). In a discrete space, e.g. the two dimensional square lattice, this means adding the value of \( f \) in the nearest neighbours of the site \((x,y)\) and subtracting four times the value of \( f \) at the site \((x,y)\) itself. It is these coefficients that are summarised in the matrix \( \Delta \) (1.1).
This ensures that no grains are created in the toppling process.

The introduction of the sink site has the advantage that the matrix $\Delta$ has a nicer structure, with all diagonal elements equal to $z$, the coordination number. The row and column corresponding to the sink site do usually not appear in the matrix.

The dynamics of the BTW model (1.2) is illustrated in figure 1.2 on a simple graph with six vertices (which is, as we will see later on, a Sierpinski gasket of generation 1).

### 1.4.2 The Manna sandpile model

In 1990 S. Manna introduced an alternative sandpile-like model [19]. The Manna model differs only in a few points from the BTW model, namely:

- in a stable state each site can be occupied by at most one grain of sand, so $z_{ci} = 1$ for all sites in the system;

- the toppling rule is that two grains of sand are randomly distributed to distinct neighbours.

The major difference with the BTW model is that the distribution rule is stochastic here, while it is deterministic in the BTW case. There exist variations of the Manna model where the two grains of sand are allowed also to be transmitted to one and the same neighbour. It is believed that such a choice does not influence the critical behaviour.

Models such as the BTW model and the Manna model are not 'models' in the original sense of the word, they do not stand model for some specific physical system. They are paradigmatic models rather, grasping the very basic ingredients in a large class of physical systems. They are like the pendulum in classical mechanics, like potential wells in quantum mechanics, or like the Ising model in equilibrium statistical mechanics. In this view it can be argued that the stochasticity introduced in the Manna model stands for a degree of randomness, or noise, in the energy diffusion in a system.

In the Manna model, it is observed in computer simulations that the system self-organises into a critical state characterised by power law distributions [19].

The dynamics of the Manna model is illustrated on a Sierpinski gasket of generation 1 in figure 1.3.
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Figure 1.2: Illustration of the BTW dynamics on a Sierpinski gasket of generation 1. Here the three cornersites are the only boundary sites. The numbers indicate the number of sand grains at each site.

Figure 1.3: Illustration of the Manna dynamics on a Sierpinski gasket of generation 1.
1.5 Applications of sandpile models

As explained above both the BTW and Manna sandpile models were constructed as mere conceptual models in order to try to understand how self-organised criticality is established. The BTW model represents an oversimplified sandpile dynamics and stays far from the description of real granular piles. However various systems in Nature are believed to be possibly described by sandpile-like dynamics. E.g. in the description of solar flare dynamics [13] it has been proposed that magnetic reconnection events spread energy throughout the solar corona in a sandpile-dynamical way.

However in real systems mostly a continuous description of the diffusion and dissipation is required. These continuous models are mostly referred to as OFC-type models, after the model by Olami, Feder and Christensen [4]. This model was introduced to capture the very basics of the dynamics of tectonic plate movement and Earthquakes. Also the Zhang model [20] is a sandpile-like model where the energy transport occurs in a continuous way and therefore this model is also more realistic than the discrete BTW and Manna models.

The inherent timescale separation in sandpile and other models for self-organised criticality is naturally found in many real systems. For example, the duration of an energy discharge in an earthquake is much smaller than the time it takes for this energy to build up by movement of crustal plates. There is some criticism that SOC is not really ‘self-organised’ because a parameter indeed has to be fine tuned, namely the ratio of the timescales of energy discharge and accumulation has to be close to zero. However in many real systems such a finetuning is already present.
Chapter 2

Auxiliary tools in sandpile description: graphs, discrete physics, and scaling

2.1 Basic ideas from graph theory

Our intention is to give a quite general review of existing sandpile theory and adapt and extend it for the description of sandpile models on fractal lattices. A brief review of relevant concepts and topics in graph theory is indispensable in such a review.

2.1.1 Basic nomenclature

Let us start with presenting some basic nomenclature of graph theory.

- A simple graph $G$ is a nonempty finite set $V(G)$ of vertices and a finite set $E(G)$ of edges (an edge is a connection between two vertices).

- Two vertices $i, j \in V(G)$ are called adjacent if they are connected by an edge $(i, j) \in E(G)$.

- An edge $(i, j) \in E(G)$ that connects the vertices $i$ and $j$ is said to be incident on as well $i$ as $j$.

- A subgraph $G_1$ of $G$ is a nonempty set of vertices $V_1(G_1)$ and a set $E_1(G_1)$ of edges such that $V_1(G_1) \subseteq V(G)$ and $E_1(G_1) \subseteq E(G)$, and such that each edge in $E_1(G_1)$ connects two vertices of $V_1(G_1)$. 
Two graphs $G_1$ and $G_2$ are distinct when there is no vertex occurring in both $V(G_1)$ and $V(G_2)$ and no edge occurring in both $E(G_1)$ and $E(G_2)$. When a graph can be expressed as the union of $k$ distinct graphs, it is said to have $k$ components.

A graph is connected when it cannot be expressed as the union of distinct graphs, so then it is a 1-component graph.

A graph is finitely ramified if it can be disconnected by cutting a finite number of bonds.

A set of edges of the form $(\nu_0, \nu_1) (\nu_1, \nu_2) \ldots (\nu_{m-1}, \nu_m)$ is called a walk on the graph. The walk is said to visit the vertices $\nu_0, \nu_1, \ldots, \nu_m$.

If all edges are distinct the walk is called a trail.

A trail in which all vertices are visited only once (except possible $\nu_0 = \nu_m$) is a self-avoiding path.

A self-avoiding path in which $\nu_0 = \nu_m$ is called a cycle.

A forest is a graph that contains no cycles.

A connected forest is a tree.

**Spanning trees**

An interesting subgraph, called spanning tree, is obtained from a connected graph on $G$ by deleting edges such that the subgraph remains connected but contains no cycles. Hence it is a tree that visits all vertices in the graph. For a nonconnected graph with $k$ components, we talk of a $k$-component spanning tree if this procedure is performed on each of the $k$ components.

An example of a spanning tree as well as a two-component spanning tree on a Sierpinski gasket is presented in figure 2.1.

There is a nice theorem that can be of use in calculating the number of spanning trees that can be constructed on a graph. Let $G$ be a connected simple graph with vertex set $\nu_1, \ldots, \nu_N$. Then we can define an $N \times N$ matrix $M$ with the following elements:

$$M_{ij} = \begin{cases} 
-1 & \text{if } i, j \text{ are adjacent} \\
\deg(\nu_i) & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}$$

(2.1)
with \( \deg(v_i) \) the degree of \( v_i \), i.e. the number of edges incident on the vertex \( v_i \).

Then this important theorem can be proved [23]:

**Theorem 2.1** The number of spanning trees on \( G \), denoted \( S(G) \), is equal to the cofactor of any element of \( M \):

\[
S(G) = \text{cof}_i(M) \quad \forall i.
\] \hspace{1cm} (2.2)

**Planar graphs and Euler’s formula**

An important category of graphs are planar graphs. These are graphs that can be drawn in the plane \( \mathbb{R}^2 \) without crossings (except at the vertices of course). We define a *face* of the graph as a region in \( \mathbb{R}^2 \) in which every point can be reached from every other point in that region without crossing an edge of the graph.

We mention the famous *Euler’s formula* applicable to planar graphs.

**Theorem 2.2** (Euler’s formula) In a planar graph with \( v \) vertices, \( e \) edges, \( f \) faces and \( k \) components the following relation holds:

\[
v - e + f = k + 1.
\] \hspace{1cm} (2.3)

(There exists an appropriate generalisation of this formula for nonplanar graphs, see [23].)
2.1.2 Resistor networks

Some important results of graph theory are often referred to in the context of resistor networks. Then the vertices of the graph \( G \) are called nodes and each edge \((i, j)\) is a wire which carries a resistor, \( R_{ij} \). We will continue with the case that the graph is connected and consists out of \( N \) vertices. Let a current \( I \) flow into node \( k \) and leave through the node \( l \). The problem that we want to solve is calculate the effective resistance \( R_{kl} \) between the nodes \( k \) and \( l \):

\[
R_{kl} = \frac{V_k - V_l}{I}.
\]

For this we need to find the voltage distribution \( \{V_i, i \in V(G)\} \) in the network. Since this problem is of importance for the study of waves of topplings in the BTW model (section 3.2), we will solve it in detail.

Denote \( I_i \) the net current in a node \( i \). Then we have

\[
I_i = I (\delta_{ik} - \delta_{il}).
\]

Kirchhoff’s first law states

\[
I_i = \sum_{j \neq i} x_{ij} (V_i - V_j)
\]  

(2.4)

with \( x_{ij} = 1/R_{ij} \).

By defining an \( N \times N \) matrix \( A \)

\[
A_{ij} = \begin{cases} 
-x_{ij} & \text{if } i \neq j \\
\sum_{m \neq i} x_{im} & \text{if } i = j
\end{cases}
\]  

(2.5)

(2.4) can be simplified as

\[
I_i = \sum_{j=1}^{N} A_{ij} V_j.
\]

(2.6)

The symmetric matrix \( A \) can be diagonalised, but as the sum of the elements in any row or column is zero, \( \det(A) = 0 \). This implies that one of the \( N \) equations (2.6) is redundant. Let us eliminate the equation in which
\[ i = n, \ n \text{ being some referention node for the potential. Then the remaining set of equations is (using } \sum_{j=1}^{N} A_{ij} = 0): \]

\[
I_i = I (\delta_{ik} - \delta_{it}) \\
= \sum_{j=1}^{N} A_{ij}V_j \\
= \sum_{j=1}^{N} A_{ij}V_j - \sum_{j=1}^{N} A_{ij}V_n \\
= \sum_{j=1}^{N} A_{ij} (V_j - V_n)
\]

concluding

\[
I (\delta_{ik} - \delta_{it}) = \sum_{j \neq n} A_{ij} (V_j - V_n). \tag{2.7}
\]

This set of equations (2.7) can now be solved. If we denote by \( A^{(n)} \) the matrix obtained by deleting the \( n \)-th row and the \( n \)-th column from \( A \), the solution of the set becomes:

\[
V_m - V_n = \sum_{p \neq n} \left[ A^{(n)} \right]_{mp}^{-1} I (\delta_{pk} - \delta_{pl}) \tag{2.8}
\]

or

\[
V_m - V_n = \left( \left[ A^{(n)} \right]^{-1}_{mk} - \left[ A^{(n)} \right]^{-1}_{ml} \right) I
\]

for the case \( n \neq l \), and for the case \( n = l \), (2.8) becomes

\[
V_m - V_l = \left[ A^{(l)} \right]^{-1}_{mk} I. \tag{2.9}
\]

Putting \( m = k \) in (2.9) we finally find the solution to the effective resistance-problem for the case that \( l \) is the reference node for the potential:

\[
R_{kl} = \left[ A^{(l)} \right]^{-1}_{kk} \tag{2.10}
\]

\[
= \frac{\text{det}A^{(kl)}}{\text{det}A^{(l)}} \tag{2.11}
\]
where $A^{(kl)}$ is the $(n-2) \times (n-2)$ matrix obtained by deleting both the $k$-th and $l$-th row and column from $A$.

In the case a site $n \neq l$ is the reference node for the potential, the expression for $R_{kl}$ becomes a little more complicated. In that case the potential difference between sites $k$ and $l$ is

$$(V_k - V_n) - (V_l - V_n) = \left( \left( \left[ A^{(n)} \right]^{-1}_{kk} - \left[ A^{(n)} \right]^{-1}_{kl} \right)
- \left( \left[ A^{(n)} \right]^{-1}_{lk} - \left[ A^{(n)} \right]^{-1}_{ll} \right) \right) I$$

or

$$V_k - V_l = \left( \left[ A^{(n)} \right]^{-1}_{kk} + \left[ A^{(n)} \right]^{-1}_{ll} - \left[ A^{(n)} \right]^{-1}_{kl} - \left[ A^{(n)} \right]^{-1}_{lk} \right) I,$$

and the effective resistance is

$$R_{kl} = \left[ A^{(n)} \right]^{-1}_{kk} + \left[ A^{(n)} \right]^{-1}_{ll} - \left[ A^{(n)} \right]^{-1}_{kl} - \left[ A^{(n)} \right]^{-1}_{lk}. \ (2.12)$$

However the resistance $R_{kl}$ is merely a property of the network and should not depend on the choice of the reference node for the potential. Therefore (2.12) coincides with (2.10).

It is possible to relate this result to properties of spanning trees, in the case that all resistors are of unit size ($R_{ij} = 1/x_{ij} = 1$). Then the matrix $A$ used here coincides with the matrix $M$ defined in the previous section. Clearly, following theorem 2.1, $\det \left( A^{(l)} \right) = S(G)$, where $G$ is the graph associated with the resistor network. In graph theory it can be shown [23] that $\det \left( A^{(kl)} \right) = S_{kl}(G)$, the number of two-component spanning trees where $k$ and $l$ are in different components. With these results the effective resistance can be expressed as

$$R_{kl} = \frac{S_{kl}(G)}{S(G)}. \ (2.13)$$

2.1.3 Relationship to the Potts model

The number of spanning trees on a graph can be calculated using the Potts model. This relationship is highly interesting, because it links graph theory and statistical mechanics. Since we will see that some properties of the sandpile model can be related to graph theory, it will be possible to calculate these properties using statistical mechanical tools.
The relation between the Potts model and graph theory was originally discovered by Fortuin and Kasteleyn [24].

In the Potts model a spin variable $\sigma_i$ is assigned to each vertex $i$ of a connected graph $G$. $\sigma_i$ can assume $q$ integer values: $\sigma_i \in \{1, \ldots, q\}$. The reduced Hamiltonian for the Potts model on the graph $G$ is

$$-\mathcal{H} = K \sqrt{q} \sum_{(i,j) \in E(G)} \delta_{\sigma_i, \sigma_j}$$

The corresponding partition function is

$$Z(K, q) = \text{Tr}_{\{\sigma_i, i \in V(G)\}} e^{\mathcal{H}}.$$

The extra factor $\sqrt{q}$ in $\mathcal{H}$ is in principle not necessary, but in order to calculate the number of spanning trees, it has to be there. The principle of the calculation is to calculate $Z$ in the limit $q \to 0$, and the factor $\sqrt{q}$ is necessary to obtain a limit that can be related to the number of spanning trees.

We start the calculation by noticing

$$e^{K \sqrt{q} \delta_{\sigma_i, \sigma_j}} = 1 + \delta_{\sigma_i, \sigma_j} p(K, q)$$  \hspace{1cm} (2.14)

where

$$p(K, q) = e^{K \sqrt{q}} - 1.$$  \hspace{1cm} (2.15)

Using this, $e^{\mathcal{H}}$ can be written as:

$$e^{\mathcal{H}} = \prod_{(i,j) \in E(G)} \left[ 1 + \delta_{\sigma_i, \sigma_j} p(K, q) \right].$$  \hspace{1cm} (2.16)

Let $e(G)$ be the number of edges in $G$. The product on the r.h.s. of (2.16) contains $2^{e(G)}$ terms. For each of these terms, $E(G)$ can be divided into two disjoint sets: one containing the edges that appear in the index of the Kronecker $\delta$'s that are present in the term, these edges contribute a weight factor $p(K, q)$. And the other set are the remaining edges, which have a weight factor 1. Each term can thus be presented by a subgraph $G'$ of $G$. The vertex set of $G$ and $G'$ are the same, but $G'$ contains only those edges that appear in the Kronecker $\delta$'s of the specific term. We denote by $e(G')$ the number of edges in $G'$. (2.16) can then be written as:

$$e^{\mathcal{H}} = \sum_{G'} p(K, q)^{e(G')} \prod_{(i,j) \in E(G')} \delta_{\sigma_i, \sigma_j}$$  \hspace{1cm} (2.17)
where the sum runs over all subgraphs \( G' \) of \( G \) such that \( V(G') = V(G) \) and \( E(G') \subset E(G) \).

To calculate \( Z \) we have to take the trace of \( e^{H} \) over all values of all spins. Let the graph \( G' \) have \( k(G') \) components. The product over the Kronecker \( \delta \)'s in (2.17) will only be nonzero when the spins take on the same value on each vertex belonging to the same component. So when performing the trace each component will contribute a factor \( q \) to the product in (2.17). The prefactor of the product is clearly not dependent on the spins and can be put in front of the trace. The result is:

\[
Z(K, q) = \sum_{G'} p(K, q)^{e(G')} q^{k(G')} .
\]  
(2.18)

The crucial step now is to take the limit \( q \to 0 \). This limit may seem senseless in the context of the Potts model. However, it can be seen as an analytical continuation on the general Potts result (2.18). In the limit \( q \to 0 \), we can restrict \( p(K, q) \) to the lowest order term in \( q \), namely \( \sqrt{q}K \).

\[
\lim_{q \to 0} Z(K, q) = \lim_{q \to 0} \sum_{G'} K^{e(G')} \sqrt{q}^{e(G')} q^{k(G')} 
= \lim_{q \to 0} \sum_{G'} K^{e(G')} q^{(e(G') + 2k(G'))/2} .
\]  
(2.20)

Now we use Euler's formula (2.3) to eliminate \( e(G') \)

\[
e(G') = v(G') + f(G') - k(G') - 1
\]
so (2.20) becomes

\[
\lim_{q \to 0} Z(K, q) = \lim_{q \to 0} \sum_{G'} K^{v(G') + f(G') - k(G') - 1} q^{v(G') + f(G') - 1 + k(G')} .
\]  
(2.21)

By definition it is clear that \( v(G') = v(G) \). This number is fixed for a given graph, independent of the set \( E(G) \) of edges. As \( q \to 0 \), the dominant term in (2.21) will be that in which \( f(G') + k(G') \) is minimal. Since for any graph the minimal value of \( f \) as well as \( k \) is 1, the minimal value of their sum is 2, and it is only reached in the case that both \( f \) and \( k \) are 1. Thus only the subgraphs \( G' \) of \( G \) for which \( f(G') = k(G') = 1 \) are of importance in the limit \( q \to 0 \), and these subgraphs are exactly the spanning trees of \( G \). (2.21) then results in

\[
\lim_{q \to 0} Z(K, q) = \lim_{q \to 0} K^{v(G)-1} q^{(v(G)-1)/2} S(G).
\]  
(2.22)
2.1. BASIC IDEAS FROM GRAPH THEORY

This is the important relation between the Potts model and the spanning trees. Calculating the partition function \( Z(K, q) \) for the \( q \)-state Potts model on a given graph, and taking the limit \( q \to 0 \), yields an expression that is proportional to the number of spanning trees on the graph.

In the derivation, we used Euler’s formula for planar graphs (2.3). However, using the variant for nonplanar graphs yields a comparable result [23].

Two component spanning trees

Another relationship between the Potts model and graph theory can be derived, namely for the calculation of the number of two component spanning trees on a certain graph. This will be of importance in the theory of waves in the BTW model, see section 3.2.

The number of two component spanning trees with vertices \( k \) and \( l \) in separate components can be calculated from the Potts model by considering the two-point correlation function

\[
\langle \delta_{\sigma_k, \sigma_l} \rangle_G = \frac{\text{Tr}_{\{\sigma_i, i \in V(G)\}} [\delta_{\sigma_k, \sigma_l} e^H]}{Z(K, q)}
\]

In the same way as above we can write

\[
\delta_{\sigma_k, \sigma_l} e^H = \delta_{\sigma_k, \sigma_l} \prod_{(i,j) \in E(G)} \left[1 + \delta_{\sigma_i, \sigma_j} p(K, q) \right]
\]

(2.23)

\[
= \sum_{G'} p(K, q)^{e(G')} \delta_{\sigma_k, \sigma_l} \prod_{(i,j) \in E(G')} \delta_{\sigma_i, \sigma_j}.
\]

(2.24)

When performing the trace over the spin variables, in order to calculate \( \langle \delta_{\sigma_k, \sigma_l} \rangle_G Z(K, q) \), it is convenient to distinguish between the set of subgraphs \( G_1 \), containing those subgraphs where \( k \) and \( l \) are in the same component, and \( G_2 \), with those subgraphs where \( k \) and \( l \) belong to different components:

\[
\delta_{\sigma_k, \sigma_l} e^H = \sum_{G' \in G_1} p(K, q)^{e(G')} \delta_{\sigma_k, \sigma_l} \prod_{(i,j) \in E(G')} \delta_{\sigma_i, \sigma_j}
\]

\[
+ \sum_{G' \in G_2} p(K, q)^{e(G')} \delta_{\sigma_k, \sigma_l} \prod_{(i,j) \in E(G')} \delta_{\sigma_i, \sigma_j}.
\]

In the calculation for the ordinary spanning trees, we saw that each component of a subgraph \( G' \) contributes a weight factor \( q \) when performing the trace. For the current case, it is clear that nothing changes when \( k \) and \( l \) are
in the same subgraph. In that case the factor $\delta_{\sigma_k,\sigma_l}$ will appear twice in the term in (2.24) corresponding to the specific subgraph $G'$, and $\delta^2_{\sigma_k,\sigma_l} = \delta_{\sigma_k,\sigma_l}$. But in the subgraphs where $k$ and $l$ belong to different components, the factor $\delta_{\sigma_k,\sigma_l}$ eliminates one factor $q$ since two components of $G'$ will have to take on the same value.

The result is

$$
\langle \delta_{\sigma_k,\sigma_l} \rangle_{G} Z(K, q) = \sum_{G' \in G_1} p(K, q)^{e(G')} q^{k(G')} + \sum_{G' \in G_2} p(K, q)^{e(G')} q^{k(G')} - 1.
$$

Now we take the limit $q \to 0$, and thus $p(K, q) \to \sqrt{q} K$,

$$
\lim_{q \to 0} \langle \delta_{\sigma_k,\sigma_l} \rangle_{G} Z(K, q) = \lim_{q \to 0} \left[ \sum_{G' \in G_1} K^{e(G')} q^{\frac{e(G') + 2k(G')}{2}} + \sum_{G' \in G_2} K^{e(G')} q^{\frac{e(G') + 2k(G') - 2}{2}} \right].
$$

We use Euler's formula (2.3) to eliminate $e(G')$,

$$
e(G') = v(G') + f(G') - k(G') - 1
$$

and (2.25) becomes

$$
\lim_{q \to 0} \langle \delta_{\sigma_k,\sigma_l} \rangle_{G} Z(K, q) = \lim_{q \to 0} \left[ \sum_{G' \in G_1} K^{v(G') + f(G') - k(G') - 1} q^{\frac{v(G') + f(G') - 1 + k(G')}{2}} + \sum_{G' \in G_2} K^{v(G') + f(G') - k(G') - 1} q^{\frac{v(G') + f(G') - 3 + k(G')}{2}} \right].
$$

In the limit $q \to 0$ the dominant term in the first sum will be that where $f(G') = k(G') = 1$ (ordinary spanning trees), making the first sum proportional to $q^{\frac{v(G') + 1}{2}}$. It is clear that for the second sum the subgraphs must have at least two components. The dominant term there will be that for subgraphs with $k(G') = 2, f(G') = 1$. This makes the second sum proportional to $q^{\frac{v(G)}{2}}$. This term dominates over the first sum when $q \to 0$. Now, since subgraphs with $k(G') = 2, f(G') = 1$ are what we call two component spanning trees, and in the second sum $k$ and $l$ are in different components,
2.2. **THE DISCRETE LAPLACIAN**

\[
\lim_{q \to 0} Z(K, q) = \lim_{q \to 0} K^{v(G) - 1} q^{v(G)} S_{kl}(G)
\]

with \( S_{kl}(G) \) the number of two component spanning trees on \( G \) with \( k \) and \( l \) in different components.

So again we have a relation between a quantity that can be calculated for the (general) \( q \)-state Potts model and a quantity in graph theory.

Using the result for spanning trees (2.22), (2.26) can be written as

\[
\lim_{q \to 0} \sqrt{q} K^{\langle \delta_{\sigma_k, \sigma_l} \rangle_G} = \frac{S_{kl}(G)}{S(G)}.
\]

and using (2.13) this can be related to the resistor network problem, yielding

\[
R_{kl} = \lim_{q \to 0} \sqrt{q} K^{\langle \delta_{\sigma_k, \sigma_l} \rangle_G}.
\]

2.2 **The discrete Laplacian**

2.2.1 **The discrete version of the Laplacian operator**

The toppling matrix of the BTW sandpile model coincides, except for the sign, with a discrete analogous of the Laplacian operator on a lattice (see the footnote on page 7). The reason for this is clearly that in the BTW model the Laplacian is there to describe the diffusion of energy (grains of sand) throughout the lattice during an avalanche. Indeed the change in time of energy content or number of sandgrains \( z_i \) at a site \( i \) is precisely given by the components of the discrete Laplacian at that site. This corresponds to an evolution described by the well known continuous diffusion equation

\[
\frac{\partial f}{\partial t} = D \Delta f.
\]

It is in fact not so strange to talk about a discrete version of the Laplacian. The Laplacian is a very important operator in physics, useful in the description of waves, diffusion, electrostatics etc. Often the equations for these problems are derived from some discretised model, e.g.

- the wave equation can be derived taking the continuity limit of the equations of motion of a linear chain of masses interconnected with springs;
• the diffusion equation can be obtained in a similar way from the master equation for a random walker.

Then the discrete version of the Laplacian naturally appears in the equations before taking the continuity limit.

The presence of Laplacian physics in the BTW model invites us to look a bit closer at the Laplacian operator and more specific its discrete version. This is what we will do in this section.

2.2.2 The spectrum of the discrete Laplacian

It is very useful to know the spectrum of the discrete Laplacian operator. Let \( \{\lambda_\alpha, \alpha = 1, \ldots, N\} \) be the set of eigenvalues of \( \Delta \), and \( \{\bar{u}_\alpha, \alpha = 1, \ldots, N\} \) the set of corresponding eigenvectors. The eigenvalue problem is

\[
\lambda_\alpha (u_\alpha)_i = \sum_j \Delta_{ij} (u_\alpha)_j
\]  
(2.27)

The sum of the elements in any row or column of the matrix is zero. This implies that there is an eigenvalue equal to zero. From the structure of \( \Delta \) it is obvious that the corresponding eigenvector is

\[
\bar{u}_0 = \frac{1}{\sqrt{N}} (1, 1, \ldots, 1).
\]  
(2.28)

Also \( \lambda_0 = 0 \) implies \( \text{det} (\Delta) = 0 \).

The entire spectrum of the Laplacian can be derived for translational invariant (Euclidean) lattices using Fourier techniques. We will present here a one dimensional example with periodic boundary conditions. In one dimension \( V(G) = \{1, \ldots, N\} \) and \( E(G) = \{(1, 2), \ldots, (N - 1, N), (N, 1)\} \). Then (2.27) becomes

\[
\lambda_\alpha (u_\alpha)_j = -(u_\alpha)_{j-1} + 2 (u_\alpha)_j - (u_\alpha)_{j-1}.
\]

Now we try as solution

\[
(u_\alpha)_j = A e^{i k_\alpha j} + B e^{-i k_\alpha j}
\]  
(2.29)

and find immediately

\[
\lambda_\alpha = 2(1 - \cos k_\alpha).
\]  
(2.30)
The boundary conditions lead to a quantisation in $k_\alpha$. As already stated we will discuss the example of periodic boundary conditions here. This requires

$$(u_\alpha)_l = (u_\alpha)_N$$

Then from (2.29) we find

$$k_\alpha = \frac{2\pi m}{N}, \quad m = 0, \ldots, N - 1.$$ 

$m > N - 1$ will not yield new eigenvalues, as can be seen from (2.30) (indeed, an $N\times N$ matrix cannot have more than $N$ eigenvalues). So for each of the $N$ values of $m$ there is a different $k$-value, and since the different $k$-values are directly related to the index $\alpha$ we can in fact replace $m$ by $\alpha$. Also from (2.28) we find $A + B = (\sqrt{N})^{-1}$. We can choose $A = B = (2\sqrt{N})^{-1}$. Finally the spectrum of the discrete Laplacian in one dimension with open boundary conditions is

$$\lambda_\alpha = 2(1 - \cos \frac{2\pi\alpha}{N})$$

(2.31)

$$(u_\alpha)_j = \frac{1}{\sqrt{N}} \cos \frac{2\pi\alpha j}{N}.$$  

(2.32)

with $\alpha = 0, \ldots, N - 1$.

For fractal lattices this technique cannot be used. Then one could think of a recursive technique based on the (discrete) scale invariance of the fractal lattice. A good inspiration for such a work may be [25] in which such a recursive technique is used to study the solutions to the Schrödinger equation on fractal lattices.

### 2.2.3 The lattice Green function

Very important for the sandpile theory is the Green function associated to the Laplacian operator on a lattice. The definition of the (two dimensional) Laplacian Green function $G(x, y)$ in continuous space is

$$\Delta G(x, y) = \delta(x - y).$$  

(2.33)

Then an equation like $\Delta f = k(x)$ can be solved as $f(x) = \int G(x, y)k(y)dy$. From (2.33) we see that the integral operator $G(x, y)$ can formally be seen as the inverse of the differential operator $\Delta$. 
In the same spirit the lattice Green function corresponding to the discrete Laplacian can be introduced as

\[ \Delta G = I \]

or

\[ \sum_k \Delta_{ik} G_{kj} = \delta_{ij}. \]  

(2.34)

Knowing the spectrum of \( \Delta \), the lattice Green function can be calculated. Formally \( [G_{ij} = (1/\Delta)_{ij}] \), so in principle \( [G_{ij} = \sum_\alpha \frac{1}{\lambda_\alpha} (u_\alpha)_i (u_\alpha)_j] \). However this notation is not very meaningful since \( \lambda_0 = 0 \). But from (2.28) we see that in the term for \( \alpha = 0 \), \( (u_0)_i (u_0)_j \) does not depend on \( i \) or \( j \). Hence we derive a meaningful expression only for the difference of two Green function components, e.g.

\[ G_{ij} - G_{kk} = \sum_{\alpha \neq 0} \frac{1}{\lambda_\alpha} [(u_\alpha)_i (u_\alpha)_j - (u_\alpha)_k (u_\alpha)_k]. \]  

(2.35)

This derivation of (2.35) is not very rigorous. A better way to derive (2.35) is by example through the random walk problem. Let \( P(i,j,t) \) be the conditional probability that a random walker is at vertex \( i \) at time \( t \) given that he started at vertex \( j \) at \( t = 0 \). The master equation for the evolution of this probability is

\[ \dot{P}(i,j,t) = -\sum_k D_{ik} P(k,j,t) \]  

(2.36)

where the sum is over all vertices \( k \), and with \( D_{ik} = -w_{ik} \) (\( i \neq k \)), \( D_{ii} = \sum_{k \neq i} w_{ik} \), and \( w_{ik} = 1/\text{deg}(i) \) if \( i \) and \( k \) are adjacent, otherwise \( w_{ik} = 0 \). When all vertices in the graph have an equal number of nearest neighbours, i.e. \( \text{deg}(i) = z \) \( \forall i \), with \( z \) the coordination number of the lattice, then clearly

\[ D = \frac{\Delta}{z}. \]

Now we introduce the Laplace transform \( G(i,j,s) \) of \( P(i,j,t) \): 

\[ G(i,j,s) = \int_0^\infty P(i,j,t)e^{-st}dt. \]  

(2.37)
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Using (2.36) we find that this Laplace transform obeys the following equation:

\[ \delta_{ij} - sG(i, j, s) = \sum_k D_{ik}G(k, j, s) \]  

which has the formal solution

\[ G(i, j, s) = \left( \frac{1}{s.1 + D} \right)_{ij} = \left( \frac{z}{zs.1 + \Delta} \right)_{ij} . \]  

(2.39)

This becomes in terms of the eigenfunctions of the Laplacian

\[ G(i, j, s) = \sum_\alpha \frac{z}{zs + \lambda_\alpha} (u_\alpha)_i (u_\alpha)_j . \]  

(2.40)

Now we see indeed that we can relate this result to the lattice Green function, namely

\[ G_{ij} = \lim_{s \to 0} \frac{G(i, j, s)}{z} . \]

However we have to take care when performing the limit \( s \to 0 \) in (2.40) since \( \lambda_0 = 0 \). But as explained before from (2.28) we see that in the term for \( \alpha = 0 \) in (2.40) does not depend on \( i \) or \( j \). Hence in the limit \( s \to 0 \) we derive a meaningful expression for the difference of two Green function components, e.g.

\[ G_{jk} - G_{ll} = \sum_{\alpha \neq 0} \frac{1}{\lambda_\alpha} [(u_\alpha)_j (u_\alpha)_k - (u_\alpha)_l (u_\alpha)_l] . \]  

(2.41)

This is the same result as we derived in (2.35) using handwaving arguments.

On Euclidian lattices \( \{ (\lambda_\alpha, \vec{u}_\alpha), \alpha = 1, \ldots, N \} \) are known and an analytical expression can be found for \( G_{jk} - G_{ll} \), e.g. for the one dimensional lattice for periodic boundary conditions using (2.31)-(2.32)

\[ G_{jk} - G_{ll} = \frac{1}{2N} \sum_{\alpha=1}^{N-1} \cos \left( \frac{2\pi \alpha j}{N} \right) \cos \left( \frac{2\pi \alpha k}{N} \right) - \cos^2 \left( \frac{2\pi \alpha l}{N} \right) . \]  

(2.42)

An expression for the two dimensional case can be found in e.g. [26], formula (7). The results in dimensions \( d > 1 \) are in fact obvious extensions of (2.42).
2.2.4 The lattice Green function and the resistor problem

As already mentioned the Laplacian is an important operator involved in many problems in physics. Then it is also not so strange to see the same is true for the Green function associated with the Laplacian. In fact the lattice Green function already appeared in a common physics problem we discussed above, namely the calculation of the effective resistance between two nodes in a resistor network, see section 2.1.2.

Indeed when all resistors are of unit size the matrix $A$ defined in (2.5) coincides with the discrete Laplacian on the graph the resistor network is defined on. Its inverse $A^{-1}$ then coincides with the lattice Green function $G$ associated with the Laplacian. Then we can write our results (2.10) and (2.12) for the effective resistance between the nodes $k$ and $l$ where the current $I$ enters and respectively leaves the network as

$$R_{kl} = G_{kk}^{(l)}$$  \hspace{1cm} (2.43)

for the case $l$ is the potential reference node, and

$$R_{kl} = G_{kk}^{(n)} + G_{ll}^{(n)} - G_{kl}^{(n)} - G_{lk}^{(n)}$$  \hspace{1cm} (2.44)

for the case a site $n \neq l$ is the reference node. We use the notation

$$G_{ij}^{(k)} = [A^{(k)}]^{-1}_{ij}.$$  \hspace{1cm} (2.45)

As already noted in section 2.1.2, these results coincide, i.e. $R_{kl}$ does not depend on the specific choice of potential reference node. These equations are of importance in the study of waves of topplings.

2.2.5 The lattice Green function and the BTW model

Dhar pointed out in [27] that $G_{ij}$ has a special interpretation in the ASM, namely that $G_{ij}$ is the expected number of topplings at site $j$ when an avalanche was initiated at site $i$. Indeed, if such an interpretation would be correct, then the total average flux of grains of sand that leaves a site $j$ during the avalanche initiated at $i$ (averaged over many avalanches) is

$$\Phi_{out,j} = G_{ij} \Delta_{jj},$$

while the total average flux into $j$ is
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\[ \Phi_{in,j} = \sum_{k \neq j} G_{ik}(-\Delta_{kj}). \]

This average incoming flux of grains of sand should equal the average outgoing flux, expressing stationarity. At the site \( i \) where an avalanche is initiated there is an extra input of one grain of sand. This can be summarised as

\[ \sum_k G_{ik}\Delta_{kj} = \delta_{ij} \]  \hspace{1cm} (2.46)

These relations only hold for bulk sites since for boundary sites the average outgoing flux exceeds the average incoming flux.

\[ \Phi_{in,j} < \Phi_{out,j} \quad j \in \text{border}, \]

The mathematical explanation that (2.46) does not hold for boundary sites is that the toppling matrix \( \Delta \) and the Green function \( G_{ij} \) in (2.46) are in fact the Laplacian and the lattice Green function with the row and column corresponding to the sink site excluded. Therefore (2.46) does not hold for sites connected to the sink site, and these are the boundary sites.

(2.46) compared to (2.34) lets us conclude that the lattice Green function \( G_{ij} \) indeed can be interpreted as the average number of topplings at site \( j \) during an avalanche initiated at site \( i \). In fact this interpretation has a more rigorous foundation using the notion of waves of topplings, see section 3.2.

This interpretation is important and can be used for the calculation of some quantities, for example the mean number of topplings \( \langle s \rangle \) in an avalanche:

\[
\langle s \rangle = \frac{1}{N} \sum_i \left( \text{average number of topplings in the lattice in an avalanche initiated at site } i \right)
= \frac{1}{N} \sum_i \left( \sum_j G_{ij} \right)
= \frac{1}{N} \sum_i \sum_j G_{ij}. \]  \hspace{1cm} (2.47)

Later on we will introduce other quantities that can be computed using the lattice Green function.
2.3 Scaling and multiscaling

2.3.1 Finite-size scaling

The idea of finite-size scaling is very important in the theory of critical phenomena. A system exhibiting criticality is described by power laws, which are a direct consequence of the scale invariance. However, one expects systems not to be absolutely scale-invariant (in the mathematical sense). I.e. there will always be a lower and an upper bound that limit the scale invariance.

In the case of the (discrete) sandpile models, the lower bound will be the size of the smallest element, e.g., an avalanche cannot be smaller that one grain of sand.

However, the upper bound is of more importance. It is clear that the area that an avalanche can cover is limited, because it cannot be larger than the area of the whole system (the number of sites in the graph). And since there is always dissipation at the border, also the size of an avalanche (total number of topplings) is limited. And hence also the duration, etc.

This cut-off at the upper bound is described by the finite-size scaling hypothesis. This not only provides a formula that describes how the cut-off is realised, but also states that the cut-off region is stipulated by a power of the linear system extent \( L \). For a distribution function \( P_L(x) \) of a quantity \( x \), the bare power law is replaced by the expression

\[
P_L(x) \sim x^{-\tau} g \left( \frac{x}{L^D} \right)
\]

where \( g \) is a function that satisfies

\[
\begin{align*}
g(y) & \rightarrow 1 \quad \text{if } y \rightarrow 0 \\
g(y) & \rightarrow 0 \quad \text{if } y \rightarrow \infty.
\end{align*}
\]

Because of finite-size scaling we not only need to fix the power \( \tau \), but also the exponent \( D \). The simplest way to fix these parameters is by making a collapse plot, namely plotting \( x^\tau P_L(x) \) as a function of \( y = x/L^D \) for different values of \( L \). In case (2.48) holds all plots should then collapse and show the function \( g(y) \). By trial and error the most appropriate values for \( \tau \) and \( D \) can be found such that the plots collapse at best. As we will see this method is not so appropriate in case of fractal lattice backgrounds, because these imply a rather irregular form of the function \( g \), combined with a log-periodic behaviour. This makes it difficult to see when the curves nicely collapse.

The main exponents in this thesis are not only the sandpile-related exponents such as those of the finite-size scaling of avalanche distribution functions. Also of importance are several exponents related not to the sandpile
model but to the lattice it is defined on, or to other models defined on the same lattice (e.g. a random walk). Many exponents of the sandpile model can be related to mere properties of the lattice or to other models defined on the same lattice. Some important ones are:

- The **fractal dimension** of the lattice, $D_f$.
  For a definition a fractal dimension, see e.g. [7, 28, 29]. In short, a simple fractal structure in a $d$-dimensional space can be quantified by covering the structure with $d$-dimensional neighbourhoods of linear size $\epsilon$. If the minimum number of such neighbourhoods needed to cover the whole structure is $N(\epsilon)$ then this number will scale as

$$N(\epsilon) \sim \left(\frac{1}{\epsilon}\right)^{D_f}.$$  

(2.50)

The fractal dimension $D_f$ derived in this way is also referred to as the **box dimension**. It is a simplified version of the more general Haussdorff dimension, see e.g. [7, 28, 29].

The number of sites $N$ in a fractal lattice of linear size $L$ can be derived from (2.50). In this case we take neighbourhoods around the sites of the lattice of linear size 1, and then $\epsilon \sim 1/L$. Then follows

$$N \sim L^{D_f}.$$  

(2.51)

- The **chemical path exponent**, $z$.
  This exponent describes the scaling of the length of a chemical path on the lattice. A chemical path is the shortest path between two lattice points on the lattice. Its length $\ell_{cp}$ scales as

$$\ell_{cp} \sim r^z$$

when $r$ is the linear distance between the two lattice points.

- The **walk dimension**, $D_{rw}$.
  This is the fractal dimension of a random walk on the specific lattice. After $t$ timesteps the mean square displacement $\langle r^2 \rangle$ of the random walkers is given by (e.g. [10])

$$\langle r^2 \rangle \sim t^{D_{rw}}.$$  

(2.52)
A first important example where some of these exponents appear is the scaling behaviour of the lattice Green function. That is assumed to show the following scaling behaviour for any length scale \( b \) [30]:

\[
G_{ij}(L) = b^{D_{ru} - D_f} \tilde{G} \left( \frac{i}{b^{D_f}}, \frac{j}{b^{D_f}}, \frac{L}{b} \right)
\]  

(2.53)

The scaling form of (2.53) is typical for two-point scaling functions obeying finite-size scaling [30].

Sandpile models are presumed critical and this implies the existence of a set of finite size scaling exponents. First we have the various avalanche distribution functions \( P(x) \) with \( x \) some property of the avalanches. For every \( x \) there is a couple of finite size exponents \((\tau_x, D_x)\). The index \( x \) in the exponents \( \tau_x \) and \( D_x \) indicates to which quantity \( x \) these exponents refer to and should in this case not be seen as independent variable. \( x \) can be: \( x \in \{a, s, t, d, \ldots\} \), where

- \( a \) = the area of the avalanche (number of distinct sites toppled)
- \( s \) = the size of the avalanche, this is the total number of individual topplings
- \( t \) = the duration of the avalanche, this is the total number of timesteps it takes for the avalanche to finish. [One has to be careful defining a timestep in a sandpile avalanche process. When we would draw the avalanche as a branching process on a \( D_f + 1 \) dimensional lattice, each layer stands for one timestep. Another way to see it is to update all supercritical sites simultaneously, this constitutes one timestep.]
- \( d \) = the dissipation exponent. During an avalanche a number \( d \) of sandgrains will be dissipated at the border sites.

Other possibilities include \( x = \ell \), the perimeter of the avalanche, or \( x = r \), the linear size of an avalanche. \( r \) can be defined in several ways, e.g. the maximum of the set of distances from every point in an avalanche to the avalanche starting point, or the mean value of this set etc.

One could also investigate spatial structure of avalanches, where in the spirit of finite size scaling one expects a behaviour

\[
a \sim r^{D_{av}}, \quad \ell \sim r^{\beta_b}
\]  

(2.54)

where \( D_{av} \) is the fractal dimension of the avalanche and \( \beta_b \) the fractal dimension of the boundary. Relations like (2.54) can be extended to any couple of avalanche properties \( x \) and \( y \) by introducing the exponents \( \gamma_{xy} \):
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\[ x \sim y^{\gamma_{xy}}. \]  

(2.55)

From probability invariance it follows that

\[ P(x)dx = P(y)dy = P(x)\frac{dx}{dy}dy = P(x)\gamma_{xy}y^{\gamma_{xy} - 1}dy \]

from which we conclude

\[ P(y) \sim y^{\gamma_{xy} - 1}P(x) \]

and using the finite size scaling behaviour of the distribution functions for \( x \) and \( y \) small enough we find the exponent relation

\[ \gamma_{xy} = \frac{\tau_y - 1}{\tau_x - 1}. \]

Also a relation between the cut-off exponents \( D_x \) and \( D_y \) can be derived since relation (2.55) should also hold for the cut-off values

\[ L^{D_x} \sim L^{\gamma_{xy}D_y}. \]

In conclusion for each set of avalanche properties \((x, y)\) we have the following exponent relations

\[ \gamma_{xy} = \frac{\tau_y - 1}{\tau_x - 1} = \frac{D_x}{D_y}. \]  

(2.56)

An interesting consequence from (2.56) is that the quantity \( D_x(\tau_x - 1) \) is an invariant, i.e. it assumes the same value for each avalanche property \( x \) within one and the same sandpile model (of course under the validity of the finite size scaling assumptions). We will denote this invariant quantity as \( \zeta \):

\[ \zeta = D_x(\tau_x - 1) \quad \forall x. \]  

(2.57)

2.3.2 Discrete scale invariance

The aim of this thesis is to study sandpile models on deterministic fractal lattices. The self-similarity property of such lattices imposes certain conditions on observables of models defined on them, such as probability distribution functions, correlation functions etc. Indeed, self-similarity is a symmetry property, and symmetry restricts the set of possible solutions for the
behaviour of these observables. For a review-and-beyond on discrete scale invariance see [31].

The self-similarity can be expressed mathematically by insisting on the homogeneity of e.g. a probability distribution function $P(x)$:

$$P(x) = \mu P(\lambda x). \quad (2.58)$$

$(2.58)$ states that under a change of scale $x \rightarrow \lambda x$, $P$ does not change except for a multiplicative factor $\mu$ (that depends on $\lambda$). For lattices exhibiting discrete scale invariance $(2.58)$ holds for one specific value of $\lambda$. Of course from $(2.58)$ it follows that

$$P(x) = \mu^k P(\lambda^k x) \quad \forall k \in \mathbb{Z}.$$

The general solution of $(2.58)$ for $P$ is a power law $P(x) = cx^\alpha$. Then $(2.58)$ leads to

$$\mu \lambda^\alpha = 1. \quad (2.59)$$

Since $1 = e^{i2\pi n}$ for arbitrary $n \in \mathbb{Z}$, the general solution of $(2.59)$ is

$$\alpha = -\frac{\log \mu}{\log \lambda} + i\frac{2\pi n}{\log \lambda}. \quad (2.60)$$

So in general probability distribution functions, correlation functions etc. in systems defined on self-similar lattices will expose a power law behaviour with a complex exponent. For $n = 0$ in $(2.60)$ we find the ordinary power law that is observed in systems at criticality on e.g. Euclidean lattices.

In case $n \neq 0$ the bare power law is modulated by a factor that represents an log-periodic modulation. Indeed this can be clearly seen by rewriting this factor as

$$x^{i \frac{2\pi n}{\log \lambda}} = e^{i 2\pi n \frac{\log x}{\log \lambda}}.$$

In a log-log plot the complex power law will appear as a tilted periodic function of which the period is $\log \lambda$.

### 2.3.3 Multifractal scaling

There are complex and interesting distributions in Nature which do not obey simple finite size scaling but a more complex form of scaling instead, namely multifractal scaling or simply multiscale. To explain this scaling behaviour it is useful to shortly explain what a multifractal is and where the difference lies with a simple fractal [28, 29].
Multifractal measures

In section 2.3.1 we very shortly discussed the box dimension of a fractal structure. In the same spirit we can now discuss multifractals.

Multifractality is in fact a property of a measure defined on a fractal. An example can be the local density at points that are fractal distributed in space, see e.g. the multifractal binominal measure [28]. An ordinary fractal can be seen as a multifractal where this measure is a constant in all points. Let \( \mu_f(\epsilon) \) denote the total measure within a neighbourhood of linear size \( \epsilon \) around the point \( \vec{r} \). In a fractal as well as a multifractal \( \mu_f(\epsilon) \) will obey a scaling behaviour

\[
\mu_f(\epsilon) \sim \epsilon^{\alpha(\vec{r})}.
\]

(2.61)

In a simple fractal structure \( \alpha(\vec{r}) \equiv \alpha \) will not depend on the location \( \vec{r} \) and hence neither will \( \mu_f(\epsilon) \equiv \mu(\epsilon) \ \forall \vec{r} \). However in a multifractal structure \( \mu_f(\epsilon) \) will depend on the location \( \vec{r} \) and may even fluctuate wildly from point to point. Let us now take for \( \alpha \) a specific value and let us look only at the set of points \( \{ \vec{r}_\alpha \} \) where

\[
\mu_{\vec{r}_\alpha}(\epsilon) \sim \epsilon^\alpha
\]

(more specifically we take the set of points \( \{ \vec{r}_\alpha \} \) which have an \( \alpha \)-value between \( \alpha \) and \( \alpha + d\alpha \). If the minimum number of neighbourhoods of linear size \( \epsilon \) needed to cover this set is denoted by \( N_\alpha(\epsilon) \), then in a multifractal structure this number will scale as

\[
N_\alpha(\epsilon) \sim \left( \frac{1}{\epsilon} \right)^{f(\alpha)}.
\]

(2.62)

In a simple fractal structure, \( \mu_f(\epsilon) \) does not depend on \( \vec{r} \), so \( \alpha \) will be the same for all points in the structure, and \( f(\alpha) \) coincides with the box dimension. Simply stated, in a multifractal structure various parts or components of the structure have very different properties. These properties are quantified by the number \( \alpha \), which is called the pointwise dimension, singularity exponent or Hölder exponent. Then the set of points which have the same singularity exponent has box dimension \( f(\alpha) \). The whole structure hence can be seen as a set of complexly intertwined fractals.

Multifractal scaling

The derivation above of the multifractal analogous to the box dimension is done for a spatial structure rather than for e.g. a distribution. In the case
of a probability distribution function \( P(x) \), \( \mathcal{F} \) can simply be replaced by the independent parameter \( x \). We should see this as follows. By e.g. a simulation of a sandpile model a long time series of the quantity \( x \) is generated. We know that, in the case of finite size scaling, \( x \) is limited by

\[
1 \leq x \leq L^{D_x}.
\]

It then seems logical to describe the "measure" \( x \) by

\[
x = L^\alpha
\]

with \( 0 \leq \alpha \leq D_x \), so \( \epsilon = L \) here.

What we need now is the minimum number of environments of size \( L^\alpha \) necessary to cover the set of that inputs \( x \) in the time series that correspond to \( x = L^\alpha \). This means in fact: for how many inputs \( x \) in the time series does \( L^\alpha \) equal \( x \)? This number \( N_\alpha(L) \) can of course be derived from \( P_L(x) \), the probability distribution function extracted from the time series or sample. \( P_L(x) \) equals the probability of finding an \( x \)-value between \( x \) and \( x + dx \).

Now \( \alpha \sim \log x \), and let us denote by \( w_L(x) \) the probability of finding a value of \( \log x \) between \( \log x \) and \( \log x + d\log x \). Then \( w_L(x) \) is the probability of finding an \( \alpha \)-value between \( \alpha \) and \( \alpha + d\alpha \), and it will scale as

\[
w_L(x) \sim L^{f(\alpha)}
\]

for a (multi)fractal structure. By invariance of probability

\[
w_L(x) d\log x = P_L(x) dx,
\]

and we conclude that \( P_L(x) \) scales as

\[
P_L(x) \sim L^{f(\alpha) - \alpha}
\]

for a (multi)fractal structure.

It may be helpful to visualise the time series as a block diagram. At each instant the input \( x \) is represented by a block of height \( x \) and unit width (or vice versa). Then an environment of size \( L^\alpha \) is a block of height \( L^\alpha \) and unit width, and then it is clear that the number of these blocks we need to cover the set of blocks with height \( x \) is proportional to \( w_L(x) \).

The set of fractal dimensions \( \{ f(\alpha) \} \) is called the multiscalar spectrum.
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As we will see in the case of finite size scaling the spectrum is entirely defined by the two parameters, $\tau_x$ and $D_x$. For structures with a more complex form of scaling, the spectrum will become more complicated. Hence the multifractal scaling picture is a logical expansion of the finite size scaling picture.

Calculating the multifractal spectrum

A rather direct and useful way to calculate the multifractal spectrum for a probability distribution function was derived by Chhabra and Jensen [32]. The spectrum is parametrised by a parameter $q \in \mathbb{R}$. Denoting $\langle h(x) \rangle_L = \int h(x) P_L(x) dx$ for a general function $h$ of a quantity $x$ with probability distribution function $P_L(x)$ in a system of linear size $L$, the parametrised equations derived in [32] are

$$\mu_{x,L}(q) = \frac{x^q}{\langle x^q \rangle_L} \quad (2.65)$$

where $\mu$ does not stand for measure but is simply a shortcut notation, and

$$\begin{align*}
\alpha_L(q) &= \frac{\langle \mu_{x,L}(q) \log x \rangle_L}{\log L} \\
\beta_L(\alpha) &= -\frac{\langle \mu_{x,L}(q) \log \mu_{x,L}(q) \rangle_L}{\log L} \quad (2.66)
\end{align*}$$

Examples of multifractal spectra

- A simple fractal
  As already mentioned in section 2.3.3 in a simple fractal $\mu(\epsilon)$ is independent of $\vec{r}$, hence only one value for $\alpha$ can appear. Since it is the fractal dimension $D_f$ that describes the scaling of the measure, $\alpha = D_f$. From (2.50) and (2.62) we find that also $f(\alpha) = D_f$. Hence the multifractal spectrum for a fractal consists only out of one point

$$\left( \alpha, f(\alpha) \right) \equiv (D_f, D_f). \quad (2.67)$$

- Finite size scaling
  $\alpha$ is defined by (2.63) and $f(\alpha)$ by (2.64). From the general form (2.48) and (2.64) it follows that

$$P_L(L^\alpha) = L^{-\alpha \tau_x} g(L^{\alpha-D_x}) \sim L^{f(\alpha)-\alpha}.$$
From the behaviour (2.49) of $g$ we find (for large $L$)

$$
\begin{cases}
0 \leq \alpha \leq D_x : f(\alpha) = -(\tau_x - 1)\alpha \\
D_x < \alpha : f(\alpha) = -\infty
\end{cases}
$$

(2.68)

The set of equations (2.68) determines the multifractal spectrum for a probability distribution function that obeys finite size scaling. It is linear and runs from the origin to the point $(D_x, -(\tau_x - 1)D_x) = (D_x, -\zeta)$. In this way the multifractal analysis yields a new method of testing finite size scaling behaviour and extracting the parameters $\tau_x$ and $D_x$ besides the collapsing plot method described in section 2.3.1.

However this method is not so satisfactory since this method is only very slowly converging, as described in e.g. [33, 34]. This means that the spectrum plotted with (2.65)-(2.66) for any $L$ reasonable attainable in numerical simulations will still be rather far from the linear form (2.68). This was nicely shown in [33] for an a priori defined form of $P_L(x)$ for which the finite-$L$ form of the spectrum can be calculated exactly.

By using extrapolation methods an $L \to \infty$-form of the spectrum can be derived, but the convergence is so slow that even this extrapolated form can still be rather far from the linear spectrum. In [33] it is empirically found that one trustful characteristic of the extrapolation method used therein is that for values of $q \geq 1$ the parametrised points (2.66) accumulate around the point $(D_x, -\zeta)$, which coincides with the endpoint of the linear part in (2.68). The general form of the multifractal spectrum in the case of finite-size scaling is illustrated in figures 2.2-2.3.

- **Multifractal scaling**

In the case of multifractal scaling $f(\alpha)$ can be anything except for the single point (2.67) or the linear form (2.68). Many forms can be found in the literature. Well known examples are the multifractal binomial measure [28], aggregation models like DLA, and fully developed turbulence in three dimensions. In [33, 35] it was found that the probability distribution function of the avalanche size in the two dimensional BTW model does not converge towards the linear form (2.68). The largest part of the spectrum seems to converge to a linear form, but there is no accumulation of points around $(D_3, -\zeta)$. Instead the spectrum continues downwards. This is an indication of multifractal scaling present in the BTW model, at least in two dimensions. We will investigate the
presence of multiscaling for the sandpile model on a Sierpinski gasket in section 4.3.4.

2.3.4 Moment analysis

A concept closely related to multifractal analysis, and which will turn out to be of excellent use in the analysis of scaling in sandpile models, is moment analysis. The $q$'th moment of a quantity $x$ is defined by

$$
\langle x^q \rangle = \int_{-\infty}^{\infty} x^q P(x) dx.
$$

(2.69)

In general one can assume that the moments will follow a scaling form

$$
\langle x^q \rangle_L \sim L^{\sigma(q)}.
$$

(2.70)

The interesting property of a moment analysis is that it can detect finite size scaling quite easily. Indeed, for a probability distribution function following finite size scaling (2.48) the scaling behaviour of the moments is rather simple. Inserting (2.48) in the general moment expression (2.69) gives

$$
\langle x^q \rangle_L \quad = \quad \int_{1}^{x_{\text{max}}} x^{q-\tau_x} g \left( \frac{x}{L^{D_x}} \right) dx
$$

(2.71)

$$
= \quad L^{D_x(q-\tau_x+1)} \int_{L^{-D_x}}^{y_{\text{max}}} y^{q-\tau_x} g(y) dy
$$

(2.72)

with $x = yL^{D_x}$. If the function $g$ is such that the integral in (2.72) remains finite then the moments will scale as (2.70) with

$$
\sigma(q) = D_x(q - \tau_x + 1)
$$

i.e. $\sigma(q)$ is a linear function. Considering the conditions (2.49) upon the function $g$ one can expect they usually will be sufficient in order to keep the integral in (2.72) finite. However problems may arise when $q - \tau_x \leq -1$ because then the integral's lower bound $L^{-D_x}$ may change the overall scaling behaviour. In that case it is useful to split the integral over a domain $[L^{-D_x}, y^*]$ in which $g \approx$ constant, and the remaining domain. Now it is reasonable that the integral over the second domain is finite since $g \to 0$ for $y \to \infty$. The integral over the first domain is proportional to $L^{-D_x(q-\tau_x+1)}$.
\((y^*)^{q - \tau_x + 1} \sim L^{-D_x(q - \tau_x + 1)}\) in the case \(q - \tau_x < -1\). This annihilates the overall scaling behaviour. In the case \(q - \tau_x = -1\) the integral results in a logarithmic correction factor to the overall scaling behaviour.

In conclusion the moment scaling behaviour in the case of finite size scaling is described by

\[
\sigma(q) = \begin{cases} 
D_x(q - \tau_x + 1) & \text{when } q > \tau_x - 1 \\
0 & \text{when } q \leq \tau_x - 1
\end{cases} \quad (2.73)
\]

Due to the logarithmic corrections near \(q = \tau_x - 1\) deviations from the piecewise behaviour can be expected in this region for systems of small size. In general a fast convergence of the slope of \(\sigma(q)\) towards a constant value for \(q > \tau_x - 1\) will be an indication of finite size scaling.

**Relation between multifractal scaling and moment analysis**

There exists a simple relationship between the multifractal spectrum \(f(\alpha)\) and the moment scaling exponent \(\sigma(q)\), namely a Legendre transform:

\[
\sigma(q) = \sup_{\alpha} \{f(\alpha) + q\alpha\} \quad (2.74)
\]
\[
f(\alpha) = \inf_q \{\sigma(q) - \alpha q\}. \quad (2.75)
\]

This has been extensively shown in [33]. Using (2.75) we can formally also define the spectrum for values of \(\alpha < 0\).

In figures 2.2-2.3 we present the general multifractal spectrum \(f(\alpha)\) and the general moment scaling function \(\sigma(q)\) for the case of finite size scaling. The relations (2.74)-(2.75) can be easily qualitatively verified in the figures.
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Figure 2.2: General picture of the multifractal spectrum $f(\alpha)$ (a.) and the moment scaling function $\sigma(q)$ (b.) in the case of finite size scaling with $\tau > 1$. Important reference points are marked by dots.

Figure 2.3: General picture of the multifractal spectrum $f(\alpha)$ (a.) and the moment scaling function $\sigma(q)$ (b.) in the case of finite size scaling with $\tau < 1$. Important reference points are marked by dots.
Chapter 3

Sandpile models on fractals

This study of sandpile models on a fractal lattice is inspired by the historical importance of deterministic fractal lattices in statistical mechanics. Such lattices became very popular in the eighties after the introduction of the renormalisation group theory in equilibrium statistical mechanics. A real space renormalisation involves a discrete rescaling and since these lattices exhibit discrete scale invariance it seems as though they are constructed especially for renormalisation calculations. Many properties such as critical exponents of systems defined on such lattices can be calculated exactly using the renormalisation group theory.

Although the sandpile models discussed here have been studied already on different kinds of lattices, and on some of them even solved exactly, most of the work on them done so far is on the two dimensional square lattice. Therefore some results and some procedures described so far in the literature are only valid for that case and hence no comparable results can be found in the literature for a general type of lattice. This chapter is a small effort to do so (especially for the case of waves in the BTW model). It contains both new work and existing theory. One should see it combined with the next chapter which forms an illustration of these ideas on the Sierpinski gasket fractal.

3.1 The group theoretical approach

In 1990 Deepak Dhar published an important paper [27] on what is often referred to as the group theoretical approach to the BTW sandpile automaton. This paper formed the basis of some considerable analytical work on the BTW model. This approach forms thus far the most developped theoretical work on SOC models. In this section we give a summarising overview of this work, as far as it is necessary for the work presented in this thesis.
3.1.1 Characterization of the critical state

From the BTW sandpile model an Abelian group can be defined by means of toppling operators, working on the space of stable configurations $S$. The full argumentation is done in [27]. Here we repeat only the most important results.

Clearly a configuration $C = \{z_i, i = 1, \ldots, N\} \in S$ is stable when $z_i \leq z_{ic} \forall i$. The operator $a_i$ working in the space of sandpile configurations on a configuration $C$ is defined as adding a grain of sand at site $i$, and letting the system evolve under the toppling rule (1.2). The result is the stable configuration $C' \in S$ that is reached.

$$a_i : S \rightarrow S$$

$$C \rightarrow a_i C = C'.$$

It is not so difficult to see that the order in which the unstable sites are toppled during an avalanche does not influence the resulting configuration. Also adding a grain of sand at site $i$, letting the system relax, and then adding a grain at site $j$, will result in the same configuration as when first a grain was added at site $j$ and then, after relaxation, at site $i$. This means that the operators $a_i$ ($i = 1, \ldots, N$) commute,

$$[a_i, a_j] = 0.$$

Therefore the BTW model is also referred to as the Abelian sandpile model (ASM).

The important hallmark of SOC models is that they autonomously reach a unique dynamical stationary state, independent of initial conditions. The stationary state is called the SOC state and is characterized by critical properties. Dhar mathematically characterized the critical state of the ASM in [27]. An important result he found is that not all stable configurations appear in the SOC state. Some are transient and are 'forbidden' in the SOC state. Hence the set of stable configurations $S$ can be written as the distinct union of the set of recurrent configurations $\mathcal{R}$ and of the transient or forbidden configurations $\mathcal{T}$:

$$S = \mathcal{R} \cup \mathcal{T}.$$

The SOC stationary state is characterized by an evolution within the set $\mathcal{R}$ of recurrent configurations, in which each configuration has an equal probability of occurrence in the Markovian evolution of the ASM.
Dhar also constructed a procedure to define whether a given configuration is or is not allowed in the SOC state: the burning algorithm. This uses the notion of forbidden subconfigurations (FSC’s). An FSC is any set of \( r \) sites (\( r \geq 1 \)) of which the corresponding height variables satisfy

\[
z_j \leq \sum_{i \in F, i \neq j} (-\Delta_{ij}) \quad \forall j \in F. \tag{3.1}
\]

A configuration \( C \) is allowed if it contains no FSC.

**The Burning algorithm:**

Let \( C \) be the configuration we want to check. Let \( T \) be a test set of sites.

1. First choose \( T \) to be the entire lattice.

2. Check inequality (3.1) for all sites in \( T \)
   - true for all sites? \( \rightarrow C \) **disallowed**
   - not true for some sites? These sites can not be part of a FSC. Eliminate ('burn') them.

3. The remaining sites constitute a smaller test set \( T' \). Check inequality (3.1) for all sites in \( T' \).

And so on...

Finally we end up with either:

- a finite FSC, in which all sites satisfy (3.1), and \( C \) is disallowed.

- an empty set, and \( C \) is allowed.

The burning algorithm in fact comes down to **burning all the sites whose height is strictly larger than the number of their unburnt neighbours**. This burning continues until no more sites can be burnt.

A very simple example of a FSC is on the two dimensional square lattice where two adjoint sites have both heigh one. Here it is easy to understand why the configuration cannot appear in the SOC state. If sand is added to these sites such that one of them will topple, its height will decrease while the other site's height will increase. There is no way in which the original configuration with two adjoint sites both having height one can be reached again.
3.1.2 Stationary state properties

The work of Dhar allows to retrieve some interesting properties of the ASM. Most of them are stationary state properties of the SOC state, including the number of allowed configurations in the SOC state, the height distribution in the SOC state and the height-height correlation function.

The number of allowed configurations in the SOC state

The number of allowed, or recurrent, configurations in the SOC state can be calculated [27] as

$$N_C = det(\Delta). \quad (3.2)$$

This is a result that looks formally comparable to the result (2.2) obtained in graph theory regarding the number of spanning trees. Indeed a relationship between the ASM and spanning trees can be established (see section 3.1.3).

Height probabilities

It is observed that the average - taken over an interval of time - fraction of sites with a certain height reaches a constant value in the SOC state, and the fluctuations diminish when the system size increases. Hence the height probabilities are an important indication of the stationary state.

From the sandpile theory above the probability $P(1)$ that a site has height equal to one in the SOC state can be derived quite simply. This was pointed out by Majumdar and Dhar in [26]. For the calculations of the other height probabilities $P(2), \ldots$ a more difficult scheme is necessary, derived by Priezzhev in [36]. We will present only the way in which $P(1)$ can be calculated on a general lattice.

Let us take a lattice $G$ with a site $i$ that has $z$ neighbours, which are denoted as $\{i_1, i_2, \ldots, i_z\}$. Since we would like to calculate the probability $P(1)$ that a site has height equal to one, let us consider an allowed configuration $C$ in which $z_i = 1$. Now consider a configuration $C'$ which is derived from $C$ by reducing the height of $i_2$, $i_3$, $\ldots$, $i_z$ by one, and consider the ASM defined on a lattice $G'$ derived from $G$ by deleting the bonds between $i$ and $i_2$, $i_3$, $\ldots$, $i_z$. For each deleted bond we decrease the maximum allowed height at the two end sites by one.

$C'$ is an allowed configuration on $G'$, so it can be burnt entirely. This means that for each site on $G$ the height is strictly greater than the number of its unburnt neighbours. But if this is true for $C$ on $G$, it also has to hold for $C'$ on $G'$. For site $i_1$, nothing changed. For sites $i_2, i_3, \ldots$, the two sides
3.1. THE GROUP THEORETICAL APPROACH

of the inequality (3.1) were decreased by one, so it still holds. For site \( i \) both sides were decreased by \( z - 1 \), so it still holds. And for all other sites in the lattice nothing changed. So \( C' \) is an allowed configuration on \( G' \), under the modified ASM.

The modified ASM has a toppling rule matrix \( \Delta^{(i)} \) defined by

\[
\Delta^{(i)} = \Delta + B^{(i)}
\]

with the matrix \( B^{(i)} \) defined as

\[
B_{i,i}^{(i)} = -(z - 1) \\
B_{\alpha \alpha}^{(i)} = -1 \quad \text{for } \alpha = 2, 3, \ldots \\
B_{i,\alpha}^{(i)} = B_{\alpha,i}^{(i)} = 1 \quad \text{for } \alpha = 2, 3, \ldots \\
B_{k,l}^{(i)} = 0 \quad \text{in all other cases.}
\]  

(3.3)

So a correspondence has been found between all allowed configurations on \( G \) with \( z_i = 1 \), and all allowed configurations on \( G' \) under the modified matrix \( \Delta^{(i)} \). In the Markovian evolution of the ASM the system will pass through a fraction \( P_i(1) \) of configurations in which site \( i \) has height equal to 1:

\[
P_i(1) = \frac{\text{number of allowed configurations on } G \text{ with } z_i = 1}{\text{total number of allowed configurations on } G}.
\]  

(3.4)

If there are \( N \) sites in the graph and the system passes through \( M \) successive configurations (with \( M \) very large) in the ASM evolution, we will finally have a collection of \( NM \) height values of which approximately \( \sum_{i=1}^{N} P_i(1)M \) will be 1. In the limit \( M \rightarrow \infty \) this means that the fraction of sites in the graph having height equal to 1 becomes

\[
P(1) = \frac{1}{N} \sum_{i=1}^{N} P_i(1).
\]  

(3.5)

The numbers \( \{P_i(1), i = 1, \ldots, N\} \) can be calculated using the correspondence derived above. The nominator of (3.4) coincides with the number of allowed configurations on \( G' \) under the modified matrix \( \Delta^{(i)} \), and this number equals \( \text{det}(\Delta^{(i)}) \). The denominator of (3.4) equals \( \text{det}(\Delta) \). Then
\[ P_1(1) = \frac{\det(\Delta^{(i)})}{\det(\Delta)} = \det(I + GB^{(i)}) \]  

(3.6)

This probability is in general site dependent since not all sites will have a comparable environment in a general lattice.

Using comparable methods schemes can be constructed to calculate the joint probability that two (non-neighbouring) sites in the lattice have height equal to one etc. [26].

### 3.1.3 Relation between the ASM and the Potts model

Besides the relationship between spanning trees and the Potts model (see section 2.1.3), there is also a relationship between spanning trees and ASM allowed configurations. This relationship was established by Majumdar and Dhar in [37]. Because it is a nice analytical result and because it is of major importance in sandpile theory, we will present the deduction of this result here for a general graph.

The matrices \( A \) (2.5) and \( M \) (2.1) from graph theory are closely related to the toppling matrix or discrete Laplacian \( \Delta \) (1.1). The results for the number of spanning trees on a graph (2.2) and the number of allowed configurations on a lattice for the ASM (3.2) are also related. They completely coincide if the graph is extended by one site that is connected to each site \( i \) with \( \sum_{j=1}^{N} \Delta_{ij} \) bonds. In the undirected and conservative ASM this means that only the boundary sites are connected to this site, by one or more bonds, making their coordination number equal to that of the bulk sites if \( \Delta_{ii} \) for all sites \( i \) was chosen to be equal to the bulk site coordination number. This site is called the sink\(^1\).

Let us denote the original graph \( G \) and the graph extended with a sink site \( G' \). Let us denote the toppling matrix of the ASM on \( G \) as \( \Delta \) and the toppling matrix of the ASM on \( G' \) as \( \Delta' \). The number of ASM allowed configurations on \( G \) is

\[ N_C(G) = \det(\Delta) \]

\(^1\)The sink site is mainly of graphical importance in sandpile theory. In the BTW dynamics, it can not be allowed to topple, otherwise there would be no dissipation of grains.
while the number of spanning trees on the extended graph $G'$ is

$$S(G') = cof_i(\Delta') \quad \text{for any } i.$$ 

By construction of $\Delta'$

$$cof_i(\Delta') = det(\Delta) \quad \text{for any } i.$$

so we can conclude that the number of ASM allowed configurations on a graph $G$ is the same as the number of spanning trees on the extended graph $G'$:

$$N_C(G) = S(G').$$ \hfill (3.7)

In fact it can be shown that there is an explicit one-to-one correspondence between ASM allowed configurations on the graph $G$ and the number of spanning trees on the extended graph $G'$, by using the burning algorithm (BA). Applying the BA, an allowed configuration can be burnt entirely. The trick is to burn an allowed configuration in such a way that the 'fire's path' follows a spanning tree. It can be shown that each different configuration leads to a unique spanning tree. From result (3.7) it follows that the set of allowed configurations on $G$ has the same number of elements as the set of spanning trees on $G'$. This establishes a one-to-one correspondence (bijection) between all allowed configurations on $G$ and all spanning trees on $G'$.

We will briefly show how burning an allowed configuration can lead to a unique spanning tree. Start with an allowed configuration $C$ on $G$, and add a sink site 0. Now first burn site 0 (irrespective of the BA rules), and in the next time step burn all sites that are burnable parallel (by a burnable site we mean that it can be burnt under the burning algorithm). Then repeat this parallely until the lattice is entirely burnt. The fire will follow a specific path on the lattice. Configurations differ in height (at least in one site), so for different configurations the fire will follow a different path, because the higher a site's height, the sooner it becomes burnable. There is only a problem when a site becomes burnable just after two or more of its neighbours were burnt simultaneously. Then it is possible that configurations with different height at that site will result in an equal fire path.

But it is possible to overcome this problem by appointing one of its neighbours - which one depending on the unburnable site's height - so that we can say that the fire reaches the site through that specific neighbour. Therefore
we have to record for each site in the graph a specific order of its neighbours. E.g. in a two dimensional square lattice: (N, E, S, W), indicating the neighbour in the north, east, south or west direction. This order has not necessarily to be the same for each site in the lattice.

Assume that the unburnable site \( i \) has \( K \) burnable neighbours. Suppose that after burning these neighbours, \( i \) becomes burnable, then we have by definition

\[
z_i > \xi_i
\]

with \( \xi_i \) the number of the remaining unburnt neighbours of \( i \),

\[
\xi_i = \sum_{j \neq i}^{'} (-\Delta_{ij}).
\]

where the prime indicates a sum only over all unburnt neighbours. At the previous instant \( i \) was not yet burnable, so

\[
z_i \leq \xi_i + K.
\]

So there is some number \( s \) such that

\[
z_i = \xi_i + s
\]

Now if we say that the fire reaches \( i \) through the \( s' \)th neighbour (in the specific recorded order of neighbours for site \( i \)), then we have a fire path that depends uniquely on the height of site \( i \).

Moreover with this procedure we eliminate the possibility of loops in the fire’s path, and because the fire visits all sites its path is a spanning tree (on \( G' \), since the fire starts at the sink site).

In this way a relationship between ASM allowed configurations and spanning trees has been established, and hence also between the ASM and the \( q \to 0 \) Potts model. This is quite remarkable since the Potts model is a model of equilibrium statistical mechanics, and the ASM is a nonequilibrium model.
3.2 Waves of topplings

In [38] Dhar and Manna introduced the concept of inverse avalanches as an attempt to describe the dynamical aspects of avalanches. Later Ivashkevich, Ktitarev and Priezzhev [39] introduced the concept of waves of topplings in the same spirit. Moreover they proved that waves and inverse avalanches are nothing but the same identities. Researchers mostly preferred to use the concept of waves over inverse avalanches, probably because waves describe the evolution of avalanches more in a natural, forward way while inverse avalanches describe this evolution backwards, starting from the configuration reached when the avalanche is over.

However the story of waves starts with the paper [38] in which Dhar and Manna in fact demonstrate that the first inverse avalanche can be represented by a tree structure on the lattice. Ivashkevich, Ktitarev and Priezzhev [39] extended this graphical representation by introducing the concept of waves. What they found is that each wave in an avalanche can be represented by a two-rooted spanning tree on the lattice.

3.2.1 Definition of waves

The definition of a wave of topplings in a BTW sandpile avalanche is quite simple. It is in fact a special way of carrying out an avalanche process, made possible by the Abelian property of the model.

Suppose an avalanche was initiated at site \( i \). The avalanche spreads out in the system, and eventually site \( i \) is induced to become unstable again. Now we prevent site \( i \) from toppling and wait until all other sites in the system have relaxed, still preventing site \( i \) from toppling. When all sites have relaxed - except site \( i \) - the set of topplings that took place is called the first wave of topplings.

Now we topple site \( i \) again and start a similar procedure. The set of topplings that take place now is called the second wave of topplings. And so on, until the last wave of topplings. The configuration reached when the avalanche is over is not influenced by this special order of topplings.

At this moment it is already clear that (see [39]):

**Theorem 3.1** Denote \( F_k \) the set of sites that topple in the \( k \)'th wave and \( i \) the site where the avalanche started. Then:

1. All sites that topple in a wave topple only once in that wave.

2. The site \( i \), eventually with some other sites, undergoes the maximum number of topplings in an avalanche.
3. The site $i$ is in the bulk of each wave except for the last one, $F_{\text{max}}$, in which it is in the border of $F_{\text{max}}$.

4. The set $F_k$ with $z_i$ decreased by 1 forms a forbidden subconfiguration (FSC) of the configuration reached after the $k$'th wave.

Proof

1. Suppose a site $k$ topples in a wave the first time after its neighbour $j$. For the site $k$ to topple again all its neighbours have to topple first, by definition of the toppling matrix (1.1), hence $k$ can only topple twice if $j$ toppled at least twice. We repeat the same argument for all sites that toppled earlier than $j$, finally arriving at site $i$. But site $i$ topples by definition only once in a wave. The conclusion is that all sites that topple in a wave only topple once in that wave.

2. This is quite evident from the definition of waves and from point 1.

3. After a certain wave started, site $i$ has toppled and $z_i = 1$. In order to become again unstable during that wave, site $i$ has to recieve $\sum_{k \neq i} (-\Delta_{ik})$ grains of sand. In the uniform model, this implies that all its neighbours must topple in that wave, so $i$ is in the bulk of that wave. In the last wave site $i$ does not recieve enough grains of sand in order to topple again, which means that not all of its neighbours topple in that wave. Hence $i$ is in the border of $F_{\text{max}}$.

4. For a general site $j \neq i$ in $F_k$, its height after the $k$'th wave will be

$$z_j = z'_j - \Delta_{jj} + \sum_{m \in F_k, m \neq j} (-\Delta_{jm}),$$

$z'_j$ being its height before the $k$'th wave ($z_j = z'_j$ for sites in the bulk of $F_k$). Since the site $j$ was stable just before the $k$'th wave started, $z'_j \leq \Delta_{jj}$, or $z'_j - \Delta_{jj} \leq 0$, resulting in

$$z_j \leq \sum_{m \in F_k, m \neq j} (-\Delta_{jm}). \quad (3.8)$$

Site $i$ was not stable before the $k$'th wave started, it had one grain in excess. Therefore it is necessary to decrease $z_i$ by 1 in order for (3.8) to hold for site $i$.
3.2. WAVES OF TOPPLINGS

From (3.1) we see that (3.8) indicates that $F_k$ is a forbidden subconfiguration of the configuration reached after the $k'$th wave if $z_i$ is decreased by 1.

\[
\square
\]

3.2.2 Waves and inverse waves

First of all in this section we prove a nice correspondence between the toppling process and the burning process. In the burning process we first burn site 0 and consequently the whole lattice (for an allowed configuration). Each site has its specific burning time which is defined as the number of parallel updates of the entire lattice under the burning algorithm until the specific site is burnt.

**Theorem 3.2** In an avalanche initiated at site 0, passing $(-\Delta j_0)$ grains of sand to each site $j$, the order in which the sites are toppled coincides with the burning times of the burning process.

**Proof**

Indeed, after burning the sink site the sites which burn first are those sites $j$ for which

\[
z_j > \sum_{k \neq j \& k \neq 0} (-\Delta j_k).
\]

When starting an avalanche at 0 the sites that topple first are those for which

\[
z_j + (-\Delta j_0) > \Delta jj.
\]

Since

\[
\Delta jj = \sum_{k \neq j} (-\Delta j_k)
\]

we can write (3.9) as

\[
z_j > \sum_{k \neq j} (-\Delta j_k) - (-\Delta j_0) = \sum_{k \neq j \& k \neq 0} (-\Delta j_k).
\]

Hence the set of sites which burn first in the burning process coincides with the set of sites that topple first in the avalanche. We thus proved the
statement for burning time \( t = 1 \). Now we proceed by induction. Suppose the statement holds up to time \( t - 1 \). Denote \( B(t) \) the set of sites that have burning times less than \( t \). At burning time \( t \) those sites \( j \) will be burnt for which

\[
    z_j > \sum_{k \neq j \& k \notin B(t)} (-\Delta_{jk})
\]

(site \( j \) is not in \( B(t) \) by definition). Now in the avalanche picture those sites \( j \) will topple at time \( t \) for which

\[
    z_j + \sum_{k \in B(t)} (-\Delta_{jk}) > \Delta_{jj}
\]  \hspace{1cm} (3.11)

(all sites in \( B(t) \) have toppled only once thus far). Again using (3.10) we can rewrite (3.11) as

\[
    z_j > \sum_{k \neq j} (-\Delta_{jk}) - \sum_{k \in B(t)} (-\Delta_{jk}) = \sum_{k \neq j \& k \notin B(t)} (-\Delta_{jk}),
\]

which concludes the proof of our statement.

\[\square\]

For a recurrent configuration the burning process will burn the entire lattice. This means that in the avalanche picture all sites will topple once. As a consequence the configuration reached when the avalanche is over is identical to the original configuration.

**Corollary 3.1** An avalanche initiated at site 0, passing \( \sum_{j \neq 0} (-\Delta_{j0}) \) grains of sand to each site \( j \), results in the original configuration. Hence toppling 0 is an identity operation.

Now we come to the waves. Suppose an avalanche is initiated at site \( i \), \( z_i = \Delta_{ii} + 1 \). Now we add one extra bond, namely from the sink site 0 to \( i \). Denote the lattice with this extra bond by \( G' \) and the original lattice by \( G \). Accordingly the BTW toppling matrix \( \Delta' \) on \( G' \) differs from \( \Delta \) in four elements:

\[
    \begin{align*}
    \Delta'_{ii} &= \Delta_{ii} + 1 \\
    \Delta'_{00} &= \Delta_{00} + 1 \\
    \Delta'_{0i} &= \Delta_{0i} = -1.
    \end{align*}
\]
3.2. WAVES OF TOPPLINGS

Since we assume that the original configuration on \( G \) is allowed under \( \Delta \), the new configuration on \( G' \) is allowed under \( \Delta' \). Thus the configuration will be burnt entirely using the burning algorithm. Now we will *initiate an avalanche at the sink site*\(^2\) like in theorem 3.2 in a special way, only possible due to the Abelian nature of the model:

- First we send a grain of sand through the bond 0\(i\), inducing \( i \) to topple. By toppling, \( i \) sends a grain of sand back to 0, and for the rest \( i \) induces an avalanche on \( G' \) in the same way as it would do on \( G \). We don’t pass another grain of sand from site 0 to \( i \), so site \( i \) can topple only once (cfr. the argument in the proof of theorem 3.1 (1)). Hence the avalanche initiated at \( i \) coincides with what we call the *first wave* on \( G \).

- Then we perform the rest of the avalanche initiated at 0, by sending the other \( \sum_{j \in \{0,i\}} (-\Delta_{j0}) \) grains of sand from 0 to all its other neighbours. That will result in another avalanche that will in a way compensate the first part of the avalanche that passed through \( i \), to result in the original configuration (corollary 3.1). This second part of the avalanche hence can be called *inverse wave*.

**Corollary 3.2** In the same way we can see that first passing \( k \) grains of sand through 0\(i\) would generate the first \( k \) waves in the original model. Then passing a corresponding amount of grains of sand through all other other neighbours of 0 would induce \( k \) inverse waves. When after a wave site \( i \) becomes stable on \( G \), passing a grain through 0\(i\) on \( G' \) would not induce a wave anymore.

Using this picture we can relate waves and spanning trees. Since we start from an allowed configuration on \( G' \) it can be burnt entirely under \( \Delta' \) resulting in a spanning tree on \( G' \). Theorem 3.2 gives us an equivalence between the burning process and the toppling process. If we perform the toppling in the special way described above, the first part of the avalanche (through 0\(i\)) will result in a part of a spanning tree covering the part of \( G \) (or \( G' \)) on which the wave takes place. When deleting the bond 0\(i\), the spanning tree on \( G' \) gets disconnected in two parts. One part stands for the wave initiated at \( i \), the other covers the rest of \( G' \). Hence a wave can be represented as a *two-rooted spanning tree* on \( G' \), with \( i \) and 0 the roots for

\(^2\)By initiating an avalanche at the sink site we mean: passing \((-\Delta_{j0})\) grains of sand to each site \( j \). Remember that the sink site takes a special place in the system, e.g. it is not allowed to topple.
the two different parts. This result has to be compared to that of Dhar in which a recurrent configuration can be represented by a one-rooted spanning tree on \( G' \), 0 being the root.

This geometrical picture derived by Ivashkevich, Kitarev and Priezzhev [39] is very useful. It relates waves to two-rooted spanning trees, of which statistics are well known. If we would not look at the avalanches and see the dynamics of the BTW model only as a consecutive series of waves, then the statistics of the waves will coincide with the statistics of two-rooted spanning trees.

The hope is to relate the known statistics of waves to the unknown statistics of avalanches. This hope has so far proven to be in vain. One has to take into account the correlations between waves within avalanches. This is a tough problem. De Menech and Stella [35] showed that waves are highly correlated in the two dimensional BTW model. In this thesis we will show that this is also the case on the Sierpinski gasket. We will come to this later.

**Corollary 3.3** Since waves can be represented by a branch of a two-rooted spanning tree, they contain no holes.

### 3.2.3 Statistics of waves

First we relate waves to the lattice Green function, and then we will use the Green function behaviour to derive the statistics of the waves.

**Waves and the lattice Green function**

Using the results for the resistor problem of section 2.1.2 and its relationship to the lattice Green function of section 2.2.4 it is easy to relate waves to the lattice Green function. In the previous section we saw that a general wave on the lattice \( G \) corresponds to a two component spanning tree on the lattice \( G \), the sink site 0 being the root for one component and the site \( i \) where the wave starts being the root for the second component.

We will earth the sink site 0 and take it the reference node for the potential. We will take the current to enter through site \( i \) and leave through a site \( j \). Then using (2.43)-(2.44) we can write down the relations

\[
R_{ij} = G_{ii}^{(0)} + G_{jj}^{(0)} - G_{ij}^{(0)} - G_{ji}^{(0)}
\]

\[
R_{00} = G_{ii}^{(0)}
\]

\[
R_{0j} = G_{jj}^{(0)}
\]

On the other hand the relationship of the resistance problem with graph theory is given by (2.13) and we can write
\[ R_{ij} = S_{i\cdot j}(G') / S(G') \]
\[ R_{i0} = S_{i\cdot 0}(G') / S(G') \]
\[ R_{0j} = S_{0\cdot j}(G') / S(G'). \]  

From (3.12) and (3.13) we derive the equations

\[ G^{(0)}_{ij} + G^{(0)}_{ji} = R_{i0} + R_{0j} - R_{ij} \]
\[ = \frac{S_{i\cdot 0}(G') + S_{0\cdot j}(G') - S_{i\cdot j}(G')}{S(G')} \]  

(3.14)

(3.15)

We can now split up the number of two rooted spanning trees \( S_{i\cdot 0}(G') \) as

\[ S_{i\cdot 0}(G') = S_{ij,0}(G') + S_{i\cdot j,0}(G') \]

where \( S_{ij,0}(G') \), respectively \( S_{i\cdot j,0}(G') \), stand for the number of two rooted spanning trees with \( i \) and \( 0 \) as roots and with site \( j \) in the subtree with \( i \), respectively \( 0 \), as root. After performing such a split also in the other terms in the right hand side of (3.15), and noticing that \( S_{ab,c}(G') = S_{ba,c}(G') = S_{c,ab}(G') \) etc., (3.15) becomes:

\[ G^{(0)}_{ij} + G^{(0)}_{ji} = \frac{S_{ij,0}(G') + S_{i\cdot j,0}(G') + S_{0\cdot i,j}(G') + S_{0\cdot i,j}(G') - S_{i\cdot j}(G') - S_{i\cdot j}(G')}{S(G')} \]
\[ = \frac{S_{ij,0}(G') + S_{0\cdot i,j}(G')}{S(G')} \]

Because of obvious reasons of symmetry we conclude

\[ G^{(0)}_{ij} = \frac{S_{ij,0}(G')}{S(G')} \]

which states that the Green function component \( G^{(0)}_{ij} \) (with the sink site 0 excluded from the matrix \( \Delta \)) is proportional to the number of two rooted spanning trees on the lattice with the sink site 0 included, where the sites \( i \) and \( 0 \) are the roots of the respective trees, and where the site \( j \) is in the same subtree as \( i \). The proportionality factor is the inverse of the number of one rooted spanning trees on the lattice with 0 included.

This is once again an important relationship between physics (through the Laplacian and its inverse, \( G \)) and graph theory.

Since such two rooted spanning trees correspond to waves of topplings in the BTW model, we can now derive wave statistics.
Statistics of waves

The general result for the statistics of waves on a general (fractal) lattice was first derived by Daerden, Priezzhev and Vanderzande in [40]. Hence we would like to stress that what follows is mainly a result of our own research.

Denote \( r \) the Euclidean distance between two lattice sites \( i \) and \( j \). Putting \( b \) equal to \( r \) in (2.53) and taking the limit \( L \to \infty \), then the Green function scales as

\[
G(r) \sim r^{D_w-D_f}.
\]  

(3.16)

Since \( G_{ij} \) gives the number of two rooted spanning trees with \( i \) and \( j \) in the same subtree, it is proportional to the number of waves starting at \( i \) that at least reach \( j \). Therefore we can write

\[
G_{ij} \sim \text{Prob}_w(i \to j)
\]

(3.17)

or in terms of the linear distance \( r \) between \( i \) and \( j \)

\[
G(r) \sim \text{Prob}_w(r_w > r)
\]

(3.18)

where in (3.17) \( \text{Prob}_w \) stands for the probability that a wave starting in \( i \) reaches \( j \), and in (3.18) this is translated to the probability that the linear size \( r_w \) of a wave is at least \( r \), with \( r \) the linear distance between \( i \) and \( j \). From this we can derive the distribution function of waves. The probability that a wave has linear size \( r_w = r \) is given by

\[
\text{Prob}_w(r_w \in [r, r + dr]) = \text{Prob}_w(r_w > r) - \text{Prob}_w(r_w > r + dr)
\]

\[
\sim G(r) - G(r + dr).
\]

This probability can be written as

\[
\text{Prob}_w(r_w \in [r, r + dr]) = \tilde{P}_w(r) dr
\]

and then we find the relationship between wave statistics and the Green function:

\[
\tilde{P}_w(r) \sim \frac{dG}{dr} \sim r^{D_w-D_f-1}.
\]

Usually we want to describe waves in terms of their size \( s_w \) instead of their linear extent. Since waves contain no holes, as was proven in corollary 3.3, they are compact and scale according to the supporting lattice
\[ s_w \sim r^{D_f} \quad \text{or} \quad r \sim s_w^{1/D_f}. \]

Of course such a relation only holds for \( s_w \) large enough to avoid finite-size effects. The behaviour of the distribution function in terms of the wave size \( s_w \) for large wave sizes then follows through invariance of probability

\[
P_w(s_w) = \bar{P}_w(r(s_w)) \frac{dr}{ds_w} \sim s_w^{D_{rw}/D_f - 2},
\]

so in conclusion

\[
P_w(s_w) \quad \sim \quad s_w^{\tau_w}, \tag{3.19}
\]

with

\[
\tau_w = \frac{D_{rw}}{D_f} - 2. \tag{3.20}
\]

(3.20) is a new and very general result. It describes the large \( s_w \) behaviour of the wave distribution function. It is based only on:

- the relation between waves and two rooted spanning trees ([39] and section 3.2.2),
- the scaling behaviour of the lattice Green function (2.53), and
- the compactness of waves (corollary 3.3).

Note that (3.19)-(3.20) gives no information regarding which type of scaling the wave size probability distribution function obeys or whether it obeys scaling at all. In case it obeys finite size scaling we expect the cut-off exponent \( D_w \) to equal the fractal dimension \( D_f \) of the support because \( (s_w)_{\text{max}} = L^{D_f} \), i.e. the maximum wave size equals the lattice size since in a wave all sites can topple at most once.

### 3.2.4 Statistics of the last wave

Suppose we investigate the distribution of the last wave in each avalanche. The asymptotics of this distribution can also be determined from the knowledge of waves discussed above. This derivation was done first in [40] and hence is again part of our own research.
As explained in theorem 3.1 (3) there is a clear geometrical difference between a general wave and the last wave, this difference lies in the location of the root or initial site. In a general wave this site lies strictly in the bulk of the wave, while for the last wave this site lies on the wave's boundary.

We can try to calculate for each site in how many waves it can be in the border. This seems difficult. It is easier to turn the problem around. What we will do is to look at each wave and see for how many root locations it is a last wave. When we sum over all sites or all waves, the statistics found in each of these two ways should finally coincide. The second way is just a specific way of performing the sums that will appear in the first way. It is easy to see this if we would write down or draw the set of all possible waves. In the first way we gather all the waves in which a specific site is in the boundary, and repeat this for all sites in the lattice. In the second way we gather first the waves which geometrically coincide except for the root location, and we take out of this set all the waves with the root located in the boundary. We repeat this for each geometrically different wave.

Suppose we have \( N_b(s) \) boundary sites in a wave of size \( s \). Then we have \( N_b(s) \) possible locations for a root site so that this wave can be regarded as a last wave, or a fraction \( N_b(s)/s \) of the sites in this wave. Hence when we look at all possible waves of size \( s \), a fraction \( N_b(s)/s \) of these correspond to a last wave. Then the probability distribution function \( P_l \) for a last wave of size \( s_l \) is

\[
P_l(s_l) \sim P_w(s_l) \frac{N_b(s_l)}{s_l}.
\]  
(3.21)

Suppose the number of sites in the boundary of a wave of size \( s \) scales as (see (2.54))

\[
N_b(s) \sim R^{\beta_b} \sim s^{\beta_b}
\]  
(3.22)

with \( \beta_b \) the fractal dimension of the wave's boundary. Then, using (3.20) and (3.22), (3.21) becomes for large \( s_l \):

\[
P_l(s_l) \sim s_l^{3-\frac{D_{rw}+\beta_b}{D_f}}
\]  
(3.23)

The exponent \( \beta_b \) has been calculated exactly for a deterministic fractal by Deepak and Abhishek Dhar in [41]. The result is

\[
\beta_b = z - D_{rw} + D_f
\]  
(3.24)

Using (3.23) and (3.24) we see that the large \( s_l \) behaviour of the last wave probability distribution function is given by
3.2. WAVES OF TOPPLINGS

\[ P_l(s_l) \sim s_l^{-\gamma_l} \]  \hspace{2cm} (3.25)

with

\[ \gamma_l = 2 - \frac{z}{D_f}. \]  \hspace{2cm} (3.26)

(3.25)-(3.26) is again a new and general result. It is based on:

- the location of the root of a last wave,
- the wave boundary scaling [41], and
- results (3.19)-(3.20) for general waves.

Again (3.25)-(3.26) gives no information regarding the scaling behaviour of the last wave probability distribution function. In case it might obey finite size scaling we can expect that \( D_l \) equals \( D_f \) since the size of the last wave is limited by the system size \( s_l \leq L^{D_f} \).

**Remark: Last waves and spanning trees**

There is a relationship between the distribution of the last wave and the probability of getting a cluster of \( s \) sites disconnected from a spanning tree by deleting a bond. To see this let us remind the derivation of the graphical representation for waves of section 3.2.2. After each wave we send a grain of sand through the bond \( 0i \) in the spirit of corollary 3.2. Each time \( i \) receives a grain from 0, it topples and initiates a wave, and sends back one grain of sand to 0.

Suppose at some instant the last wave in the avalanche is initiated. By deleting \( 0i \) a subtree disconnects with \( i \) as root and this is the graphical representation of the last wave. Now, putting back \( 0i \) and sending one more grain through \( 0i \) would not trigger \( i \) to topple again. If we delete \( 0i \) now, no subtree gets disconnected and we have simply a one rooted spanning tree. This spanning tree is the graphical representation of the recurrent configuration reached when the avalanche is over. What happened when this last grain of sand was sent through \( 0i \)? In the height representation this grain of course increased \( i \)'s height by one and nothing else changed. In the graphical representation the adding of the last grain changed the graph from a two rooted spanning tree (with one of the branches standing for the last wave) to a one rooted spanning tree. In section 3.1.3 we saw that the local structure of the spanning tree only depends on the local heights. This means that the adding of the last grain at \( i \) implied the addition of one bond to
the two rooted spanning tree. This extra bond starts from \( i \) (which is logical since \( i \) is in the border of the last wave) and connects the two branches of the two rooted spanning tree. The rest of the lattice remained unchanged upon addition of this last grain.

Looking backwards at this, we notice that we can see the last wave as a subtree of the resulting spanning tree, which can be disconnected by deleting one bond from the spanning tree. As a result, the statistics of last waves will coincide with the statistics of the clusters disconnected from a spanning tree by deleting one bond. If we could calculate these latter statistics then this would yield another way of calculating the last wave distribution function \( P_l \). This looks somewhat like what Dhar and Manna did in [38] to calculate the last wave distribution on the two dimensional Euclidian lattice. However it is important to stress that it is incorrect to simply take over their calculations to among others the Sierpinski gasket.

The derivation of \( P_l \) for the two dimensional square lattice in [38] is based on results from [42]. In [38] the fact is used that on selfdual lattices the probability that a cluster of \( s \) sites gets disconnected by deleting one bond from a spanning tree coincides with the probability that a loop is formed which encloses \( s \) sites by adding a bond to the (dual) spanning tree. The latter probability was calculated in [42] and hence the former probability is also known.

It is again important to stress that this only holds on selfdual lattices. E.g. the Sierpinski gasket fractal is not selfdual. The derivation using \( N_b \) at the beginning of this section is more general. However it needs the wave boundary exponent \( \beta_b \) which has to be calculated separately. As already mentioned this problem was solved for deterministic fractals (and hence also Euclidian lattices) by Dhar and Dhar in [41].

Finally we mention that \( \beta_b \) on the selfdual two dimensional square lattice is easy to calculate. The wave boundary forms a loop on the dual lattice and since the length of such a loop is described by the chemical path dimension \( z \), \( \beta_b \) equals \( z \) on the two dimensional square lattice. This result is certainly not general, the general result is (3.24).

### 3.3 Avalanches and dynamical aspects

The dynamics of sandpile models is governed by avalanches. But it seems hard to describe them analytically. The concept of waves was introduced because they were easier to describe analytically and the hope rose that properties of avalanches could be derived from wave properties. Some efforts
3.3. AVALANCHES AND DYNAMICAL ASPECTS

in this direction have been made [43] but a clear relationship has not been found so far.

One particular type of avalanche is quite easy to describe, namely an avalanche which starts at a point in the system’s boundary. This kind of avalanche only exhibits one wave, since the initial point loses one or more grains of sand to the border, and can not regain enough grains to topple again. Therefore the statistics of this kind of avalanche coincides with the wave statistics. Since the wave statistics can be derived from the lattice Green function, the statistics of avalanches starting at the boundary can be derived from the behaviour of the lattice Green function near the boundary. We will not further discuss these boundary avalanches, but refer in this context to [44, 45].

3.3.1 Green function and avalanches

Some properties of avalanches can be calculated from the lattice Green function. In particular its interpretation from section 2.2.5 is useful. Namely, the lattice Green function component $G_{ij}$ coincides with the average number of topplings at site $j$ when an avalanche was initiated at site $i$ (on average over many avalanches). From this we can calculate the following quantities.

- **Expected number of topplings in an avalanche**

  The expected number of topplings in an avalanche is then (see (2.47))

  $$
  \langle s \rangle = \frac{1}{N} \sum_{i,j=1}^{N} G_{ij}.
  $$

  (3.27)

  This is in fact the first moment of $s$. From (2.73) we know that in the case of finite size scaling we would have

  $$
  \langle s \rangle \sim \begin{cases} 
  L^{D_s(2-\tau_s)} & \text{when } \tau_s < 2 \\
  \text{constant} & \text{when } \tau_s \geq 2
  \end{cases}.
  $$

  (3.28)

  Using the scaling behaviour of the Green function (2.53) we can derive another expression for $\langle s \rangle$. Putting as length scale $b = L$ in (2.53) we have

  $$
  G_{ij}(L) = L^{D_{rw}-D_I} \tilde{G} \left( \frac{i}{L^{D_I}}, \frac{j}{L^{D_I}} \right).
  $$

  (3.29)

  We now can use (2.51) in (3.27), and using (3.29),
\[ \langle s \rangle = L^{D_{rw}-2D_f} \sum_{i=1}^{L^{D_f}} \sum_{j=1}^{L^{D_f}} G \left( \frac{i}{L^{D_f}}, \frac{j}{L^{D_f}} \right). \]  

(3.30)

For large \( L \) hence we find the behaviour

\[ \langle s \rangle \sim L^{D_{rw}}. \]  

(3.31)

This result is quite general and doesn’t require finite size scaling of the avalanche sizes distribution. (However it is based on the scaling assumption (2.53) for the lattice Green function but this assumption is widely accepted.)

- **Expected number of waves in an avalanche**

We can also calculate the expected number of waves in an avalanche. The Green function component \( G_{ii} \) gives the expected number that site \( i \) topples when an avalanche is initiated in site \( i \) itself. From the definition of waves it is clear that this number equals the number of waves in the avalanche. Therefore the expected number of waves in an avalanche is

\[ \langle n_w \rangle = \frac{1}{N} \sum_{i=1}^{N} G_{ii}. \]  

(3.32)

We can also derive a scaling relation for \( \langle n_w \rangle \). Using (3.32), (2.51) and (2.53) we find

\[ \langle n_w \rangle \sim L^{D_{rw}-D_f}. \]  

(3.33)

This is the important relationship that explains why there are very few waves in avalanches on Euclidean lattices with \( D_f \geq 3 \) \([46]\), for on all Euclidean lattices \( D_{rw} = 2 \) and hence \( \langle n_w \rangle \) decreases with growing \( L \). (3.33) also explains the logarithmic growth of the number of waves on the square lattice. On the other hand for most fractals \( D_{rw} > D_f \), so there will be a lot of waves in a general avalanche. It is in fact not so trivial to find fractals where \( D_{rw} < D_f \). For an example, see \([47]\).
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3.3.2 Multifractality and correlations

In [33, 35, 48] it is found that on the two dimensional lattice the probability distributions for e.g. avalanche area and radius obey finite-size scaling. But the distributions of other quantities like the avalanche size or duration violate finite size scaling. For them a description in terms of a complete multifractal spectrum is necessary. This kind of behaviour is also found on the Sierpinski gasket, see section 4.3.4.

The fact that simple finite size scaling is simply not applicable in the BTW model explains the elaborate and never ending discussion in literature about the exponents of e.g. the avalanche size distribution, and connected to this the discussion on the universality classes in self-organised criticality [52, 53, 54, 55, 48]. As we will see in the case study on the Sierpinski gasket the deviations from simple finite size scaling are clear but not very large.

The small deviations from finite size scaling in these cases explains why often exponents have been found which seemed to fit the data, e.g. [17, 59].

In finding an explanation for this complex scaling behaviour, relation (3.33) is very useful. Indeed it seems thus far that only on lattices where \( \langle n_w \rangle \) grows with a power (or logarithmic) in \( L \) multifractality is found. The way in which waves are combined within an avalanche seems to be a difficult and important problem. That this problem is nontrivial can be found from wave correlations.

Let us consider a time series of waves, i.e. a sequence of wave sizes as they appear in a simulation. Like in the problem of finding the wave size distribution we will neglect the information about to which avalanche the waves belong. One can investigate wave correlations by calculating the autocorrelation \( C_w \) function in this time series

\[
C_w(t, L) = \lim_{k_{\text{max}} \to \infty} \frac{1}{k_{\text{max}}} \sum_{k=1}^{k_{\text{max}}} s_w(k + t) s_w(k) - \overline{s_w^2}.
\]  
(3.34)

\( s_w(k) \) stands for the size of the \( k \)'th wave in the time series, and further

\[
\overline{s_w} = \lim_{k_{\text{max}} \to \infty} \frac{1}{k_{\text{max}}} \sum_{k=1}^{k_{\text{max}}} s_w(k)
\]

\[
\overline{s_w^2} = \lim_{k_{\text{max}} \to \infty} \frac{1}{k_{\text{max}}} \sum_{k=1}^{k_{\text{max}}} s_w^2(k).
\]

The fact that we neglect the information to which avalanche a wave belongs is not expected to influence the nature of this correlation function.
Avalanches are more or less independent since we wait to initiate a new avalanche until the former avalanche is over. Hence waves that belong to different avalanches are also believed to be more or less independent. So \( C_w(t, L) \) can be seen as a quantity that describes the correlations of waves within avalanches.

On the basis of simulations it was proposed in [33] that \( C_w(t, L) \) follows a power law behaviour

\[
C_w(t, L) \sim t^{-\gamma_c}.
\]

Such a behaviour is confirmed on the Sierpinski gasket, see section 4.3.5.

So there are nontrivial wave correlations, presumably only on lattices with \( D_{rw} \geq D_f \). This makes it difficult to derive avalanche statistics from wave statistics.

The multifractal scaling and the wave correlation function have been studied only numerically so far ([33, 35] and in this thesis). At the moment there is no indication of how to give an analytical description.

The fact that multifractal scaling appears in the BTW model is not so interesting from the original point of view of the theory of self-organised criticality. Indeed the theory was set up to explain (self-organised) critical behaviour, of which finite-size scaling is an important hallmark.

Nevertheless multifractal scaling is also a very interesting feature in Nature. In some sense we can say that multifractality indicates an even higher degree of complexity than that in critical systems. The fact that multifractal scaling is found by De Menech and Stella [33, 35] in such a simple system as the BTW model can be regarded as an important discovery. At this moment there is no idea how to calculate the multifractal spectrum analytically. But in all its simplicity and (at least partial) tractability the BTW model may turn out to be a useful model in the process of understanding the origin of multifractality.

### 3.3.3 Consecutive wave correlations

The study of successive wave correlations began originally [49] with the objective to verify some of the assumptions in [43], a paper which attempted to derive avalanche statistics from wave statistics. Apart from this discussion, it is interesting to study consecutive wave correlations as a first step to understand the correlations between waves within an avalanche.

We begin with introducing the forward conditional probability \( P(s_{k+1} \mid s_k) \) that the \( k + 1 \)’th wave within an avalanche has size \( s_{k+1} \) given that the
previous wave had size $s_k$. Paczuski and Boettcher [49] proposed on the basis of simulations on the square lattice that this function has a scaling form

$$ P(s_{k+1} \mid s_k) \sim s_{k+1}^{-\beta} F \left( \frac{s_{k+1}}{s_k} \right) $$

(3.35)

with

$$ F(x) \sim \begin{cases} 
\text{constant} & \text{for } x \to 0 \\
\frac{1}{x^\alpha} & \text{for large } x.
\end{cases} $$

(3.36)

Next, we can also regard the backward conditional probability $\tilde{P}(s_k \mid s_{k+1})$ that the $k$'th wave in an avalanche has size $s_k$ given that the next wave will be of size $s_{k+1}$. This probability was first introduced by Hu et al. [50] and a similar scaling law was found for the square lattice:

$$ \tilde{P}(s_k \mid s_{k+1}) \sim s_k^{-\beta} \tilde{F} \left( \frac{s_k}{s_{k+1}} \right) $$

(3.37)

with

$$ \tilde{F}(x) \sim \begin{cases} 
\text{constant} & \text{for } x \to 0 \\
\frac{1}{x^{\alpha}} & \text{for large } x.
\end{cases} $$

(3.38)

In [40] and in this thesis these conditional probabilities were studied numerically on the Sierpinski gasket. A similar scaling form to (3.36) and (3.38) was found, see section 4.3.3. This is an indication that these scaling forms may be valid in general.

Let us also remark that the normalised forms of (3.35)-(3.37) are respectively

$$ P(s_{k+1} \mid s_k) \sim \left( \frac{s_{k+1}}{s_k} \right)^{-\beta} F \left( \frac{s_{k+1}}{s_k} \right) s_k^{-1} $$

(3.39)

$$ \tilde{P}(s_k \mid s_{k+1}) \sim \left( \frac{s_k}{s_{k+1}} \right)^{-\beta} \tilde{F} \left( \frac{s_k}{s_{k+1}} \right) s_{k+1}^{-1}. $$

(3.40)

In [50] it was argued that the following exponent relation should hold

$$ \bar{\beta} + \bar{\alpha} = \tau. $$

(3.41)

The relation holds for the numerical data on the square lattice [50]. It is interesting to investigate the validity of (3.41) on the Sierpinski gasket. The results are presented in section 4.3.3.
In this section we will repeat the arguments of [50] that lead to (3.41). Also some new exponent relations will be derived, namely an expression for the exponent $\beta$ and a similar relation to (3.41) between $\beta$ and $r$. As a result there will only be one independent exponent left in the set $\{\beta, r, \bar{\beta}, \bar{r}\}$.

The exponent relation (3.41) can be derived in the limit $s_k \gg s_{k+1}$. In that case, the probability function takes on the behaviour

$$P(s_k \mid s_{k+1}) \overset{s_k \gg s_{k+1}}{\sim} s_k^{-\beta - \bar{r}}. \quad (3.42)$$

Now it is important to see that in this limit

1. the root site for the wave is located 'relatively' close to the border of the $k$'th wave, and

2. the $k + 1$'th wave necessarily has a nonempty intersection with the boundary of the $k$'th wave.

**Proof.**

Within the bulk of the $k$'th wave all sites have toppled exactly once, so they return to their original height after the wave has passed. If the $k + 1$'th wave would cover a small region in the bulk of the $k$'th wave it would have to follow the motion of the previous wave since the bulk of the $k$'th wave did not change. Hence it will grow until it reaches the border of the $k$'th wave. This proves statement (2). But now statement (1) is obvious, since if the root site would be located far from the border of the $k$'th wave, the $k + 1$'th wave would have to grow quite large, and would violate the relation $s_k \gg s_{k+1}$, which was our basic assumption. With 'relatively' close we mean that the distance from the root to the border of the $k$'th wave is small enough not to violate the relation $s_k \gg s_{k+1}$. Since waves are compact this distance will be of the order, or smaller than, $r_k + 1$, the linear size of the $k + 1$'th wave, with $r_k \gg r_{k+1}$.

Then on a coarse grained scale - by performing a rescaling of the order $r_{k+1}$ - the geometry resembles that of a last wave with the $k + 1$'th wave playing the role of the origin of the avalanche, and the $k$'th wave that of the last wave. This resemblance is purely geometrical, but from it we can derive the statistics of the $k$'th wave. We derived the statistics of the last wave also on geometrical arguments, see section 3.2.4. Repeating the arguments we conclude that the $k$'th wave is distributed as

$$P(s_k \mid s_{k+1}) \overset{s_k \gg s_{k+1}}{\sim} s_k^\eta.$$
Comparing with (3.42) leads us to the exponent relation (3.41).

We can derive other exponent relations by considering the joint distribution \( P(s_k, s_{k+1}) \), this is the probability that the \( k' \)th wave is of size \( s_k \) and the \( k+1 \)'th of size \( s_{k+1} \). This distribution can be written in two ways using the conditional probabilities introduced above:

\[
P(s_k, s_{k+1}) = P(s_{k+1} | s_k) P_w(s_{k+1})
= P(s_{k+1} | s_k) P_w(s_k).
\]  

(3.43)

We then insert the respective scaling relations in (3.43) and find

\[
\left( \frac{s_k}{s_{k+1}} \right)^{-\beta} \tilde{F} \left( \frac{s_k}{s_{k+1}} \right) s_{k+1}^{1-\tau_w} \sim \left( \frac{s_{k+1}}{s_k} \right)^{-\beta} F \left( \frac{s_{k+1}}{s_k} \right) s_k^{1-\tau_w}.
\]  

(3.44)

We now insert the proper limiting behaviours of the functions \( F \) and \( \tilde{F} \). In the case \( s_k \gg s_{k+1} \) this leads to

\[
\beta = 1 + \tau_w - \tau_l.
\]  

(3.45)

Since the exponents \( \tau_w \) and \( \tau_l \) are known exactly, \( \beta \) is also known now, under the restrictions that the scaling form (3.35) should hold. When \( s_{k+1} \gg s_k \) (3.44) leads to

\[
\beta + r = 1 + \tau_w - \bar{\beta},
\]

or, by inserting (3.45),

\[
\bar{\beta} + r = \tau_l.
\]  

(3.46)

Finally (3.41) and (3.46) indicate that

\[
\bar{r} = r.
\]  

(3.47)

In conclusion, it is likely that the scaling forms (3.35) and (3.37) for consecutive waves hold for the BTW model. They are verified on the square lattice [50] and on the Sierpinski gasket [40] (and section 4.3.3). The exponent \( \beta \) is given by

\[
\boxed{\beta = 1 + \tau_w - \tau_l = 1 + \frac{z - D_{rw}}{D_f}}
\]  

(3.48)

and the exponents \( r \) and \( \bar{r} \) coincide. Finally the relation
\[ \beta + \hat{r} = \hat{\beta} + r = \tau = 2 - \frac{z}{D_f} \]  \hspace{1cm} (3.49)

leaves only one of these exponents independent.

### 3.4 The Manna model

The Manna model is more difficult to describe analytically than the BTW model. Some effort has been done by Dhar in [51], inspired by his own work on the BTW model [27]. Therefore it was necessary to discuss the problem whether the Manna model is Abelian or not. Dhar overcame this problem by insisting that each site in the model has its own local pseudo random number generator. The stochasticity in the toppling rule of the Manna model requires that the redistribution of sand depends on the outcome of a pseudo random number generator. The outside observer can not directly see a difference between the model with one general pseudo random number generator and the model with local pseudo random number generators. But the latter model is Abelian when the toppling operators depend on the state of the local pseudo random number generators. However in the Markovian evolution of the model the outcome of a toppling operator working on a sandpile state will generally not be a unique state but rather a linear combination of states.

Using this Dhar derived some analytical results partly comparable to the results for the BTW model. However in this thesis we are mainly confined to analysing the scaling properties of the Manna model and will not directly apply these results.

Another problem is finding an analogon to waves of topplings in the Manna model. If we use the basic definition of waves in the BTW model, namely preventing the avalanche initial site from toppling until all other sites have relaxed, we lose the property of theorem 3.1 (1). Indeed waves then will contain sites that have multiple topplings. But often one assumes that this property is even more important than the basic definition of waves. Then the first wave in the Manna model should be defined as the set of sites that topple in an avalanche and preventing them from toppling a second time. Then the second wave can be initiated by letting all these sites topple for the second time but preventing to topple a third time, and so on.

Clearly there is no 'natural analogon' to the concept of waves in the Manna model. Therefore we will merely investigate the statistics and scaling behaviour of avalanches for the Manna model. This is inspired by one of the major problems in sandpile theory of the last years, namely the search for and the outline of universality classes. In particular the question whether the
3.4. THE MANNA MODEL

BTW and the Manna model are in the same universality class has achieved much attention. Some papers found evidence for this, e.g. [52, 53]. Other papers disaffirmed such a conclusion, e.g. [54, 55, 48]. The conclusion was closed by the work of Stella, De Menech and Tebaldi [33, 35] that clearly showed that the two models follow a different scaling behaviour on the two dimensional square lattice, excluding a similar critical behaviour. Our work on the Sierpinski gasket yields a similar conclusion.
Chapter 4

A case study: the Sierpinski gasket

4.1 Introduction and motivation

4.1.1 Motivation

Notwithstanding impressive theoretical progress in the field - presented in chapter 3 - some basic features of sandpile models are not yet fully understood.

At the moment we can say that the stationary state properties of the BTW model are well understood: allowed configurations, height probabilities, height correlations, ... Exact results related to the dynamics of the model include the expected number of topplings in an avalanche, the expected number of waves in an avalanche, and the asymptotic properties of waves of topplings. But the dynamics of entire avalanches are not yet fully understood. The concept of waves allows us to dissect the dynamics of an avalanche into basic elements that are easier to understand. But how to combine them in order to understand avalanche dynamics? This is an important and open problem.

Studies of the wave-wave correlation function in the BTW model ([33, 35, 49] and further on in this thesis) show that waves are highly correlated, at least on lattices where $D_f < D_{rw}$. It is far from clear at the moment how to theoretically describe these correlations. In the Manna model however waves are almost uncorrelated and it is easy to derive the statistics of avalanches from the properties of waves [33].

One possible way to turn around these theoretical problems in the BTW model is to construct some dynamical renormalisation procedure on it. Some
efforts in this direction have been made so far [56, 57] but without much success, also because it is not so easy to perform an exact renormalisation on a two dimensional square lattice. This is an important reason why we decided to study sandpile models on a fractal lattice: renormalisation on self-similar lattices is often easier and can sometimes be done exactly. However the problem of finding a good dynamical renormalisation scheme for sandpile models turned out to be very hard.

But the study of sandpile models on fractals is nevertheless interesting from a broader point of view. Fractals are ubiquitous in Nature. Therefore it is important to study dynamical processes on fractal supports. Linear phenomena such as diffusion, oscillations etc. have already been studied extensively on fractals, but this is certainly not the case for nonlinear problems as sandpile models, exclusion processes etc.

Why the Sierpinski gasket? Theoretical arguments presented in section 3.3.1 show that the expected number of waves in an avalanche scales as

\[ \langle n_w \rangle \sim L^{D_{rw} - D_f} \]

with \( L \) the linear system extend. Our conjecture is that on lattices with \( D_{rw} < D_f \) the statistics of avalanches will more or less coincide with the known statistics of waves (for an indication, see e.g. [46]). Only on lattices with \( D_{rw} > D_f \) strong correlations between waves will prevent an easy extracting of avalanche statistics from wave statistics, as is the case on the square lattice.

This certainly is the case on the Sierpinski gasket where

\[ D_{rw} = \frac{\log 5}{\log 2} = 2.32192 \ldots \]

\[ D_f = \frac{\log 3}{\log 2} = 1.58496 \ldots \]

So we can expect an even more pronounced behaviour as on the two dimensional lattice. This makes the Sierpinski gasket an interesting choice.

4.1.2 Construction of the Sierpinski gasket

The Sierpinski gasket is a deterministic fractal that is constructed in the following way. Start with an equilateral triangle (modified versions in which one starts from a general triangle can also be defined). Cut out the triangle
that is formed by the middle points of each side. Then we have three upright triangles of half the linear size of the starting triangle. Within each of these triangles the same procedure is carried out. We then have nine upright triangles of one fourth of the linear size of the starting triangle. Then this procedure can be repeated again and again.

The starting triangle is said to be of generation zero. The structure obtained after the first iteration step is a Sierpinski gasket of generation one, and so on. If the iteration process would be carried out infinite times the resulting Sierpinski gasket would be a perfectly discrete self-similar structure. The Sierpinski gasket of finite generations are called prefractals.

It is simple to associate a lattice with a Sierpinski gasket of generation \( n \). The vertices or sites are simply the corner points of the smallest triangles. The edges or bonds of the lattice are the edges of these smallest triangles. It is then easy to see that the three corner points of the Sierpinski gasket have two nearest neighbour sites, while all other vertices of the lattice have four nearest neighbours.

It is appropriate to rescale the lattice after each iteration step, so that the linear size of the smallest triangle in a Sierpinski gasket of a given generation is one. Then the number of triangles increases by a factor three when doubling the linear size, from which we can conclude using (2.51) that the fractal dimension of the (in principle infinite-generation) Sierpinski gasket is

\[
D_f = \frac{\log 3}{\log 2} = 1.58496 \ldots
\] (4.1)
4.1.3 Basic properties of the Sierpinski gasket

The number of vertices $v(n)$ in a Sierpinski gasket of generation $n$ can easily be calculated using the iterative relation

$$v(n + 1) = 3v(n) - 3 = 3(v(n) - 1)$$

and the initial value $v(0) = 3$. From this recursion a closed expression for $v(n)$ follows:

$$v(n) = \frac{3}{2}(3^n + 1). \quad (4.2)$$

The linear size $L(n)$ of a Sierpinski gasket of generation $n$ is given by the formula

$$L(n) = 2^n.$$

Other geometrical quantities are easy to compute, such as the number of triangles in a gasket of generation $n$:

$$N_\Delta(n) = 3^n$$

or the number of elementary triangles (ET). By an elementary triangle we mean an object like a gasket of generation 1, represented in figure 4.1 (a). (Elementary triangle is not a standard notion but is defined ad hoc because of its use in some renormalisation procedures performed in this thesis.) The number of ET’s in a gasket of generation $n$ is

$$N_{ET}(n) = 3^{n-1}.$$

The Sierpinski gasket is finitely ramified. This means that the gasket can be disconnected by deleting a finite number of edges.

Several exponents related to the geometry of or to processes defined on the Sierpinski gasket are known exactly. The fractal dimension has already been calculated in (4.1). For large $n$:

$$v(n) \sim L(n)^{D_f}. \quad (4.3)$$

The dimension of a random walk on a Sierpinski gasket (2.52) is
4.1. INTRODUCTION AND MOTIVATION

Figure 4.2: A Sierpinski gasket of generation $n = 2$ with additional sink site indicated by $\odot$.

$$D_{rw} = \frac{\ln 5}{\ln 3} = 2.32192 \ldots$$  \hspace{1cm} (4.4)

Also the chemical path dimension $z$ of a spanning tree of a Sierpinski gasket is known exactly. This result was derived by Dhar and Dhar in [41] yielding

$$z = \frac{\log (20 + \sqrt{205})/15}{\log 2} = 1.19399 \ldots$$ \hspace{1cm} (4.5)

4.1.4 Sandpile models on the Sierpinski gasket

Once a graph is associated to the Sierpinski gasket, sandpile models can be defined on it. We can extend the Sierpinski gasket by a sink site that is connected to each of the three corner points by two edges. Then these corner points also have four neighbours, and the toppling rules for them are equal to those of the bulk sites. The sink site itself is forbidden to topple and does not really take part in the model.

Hence the toppling matrix $\Delta$ for the BTW model has the following structure. All diagonal elements are equal to four. In each row and column there
Figure 4.3: Example of site numbering in a Sierpinski gasket of generation $n = 1$.

are four elements equal to minus one, except in the three rows and columns that correspond to the corner sites, where there are only two elements equal to minus one. As usual the sink site does not show up in the toppling matrix.

As an example, with the numbering of sites as indicated in figure 4.3, the toppling matrix $\Delta$ for the first generation gasket is

$$\Delta = \begin{pmatrix}
4 & 0 & 0 & -1 & 0 & -1 \\
0 & 4 & 0 & -1 & -1 & 0 \\
0 & 0 & 4 & 0 & -1 & -1 \\
-1 & -1 & 0 & 4 & -1 & -1 \\
0 & -1 & -1 & 4 & -1 & -1 \\
-1 & 0 & -1 & -1 & 4 & -1
\end{pmatrix}. \quad (4.6)$$

In conclusion the bulk behaviour of the sandpile models on the Sierpinski gasket is not different from the behaviour on the two dimensional square lattice when one regards the toppling rule. Of course the geometry of the lattices differ and this will lead to a different behaviour. In the BTW model a corner site topples when its height is larger than four, and two grains of sand are lost. These three sites are the only sites where sand can leave the system, i.e. they are the system’s boundary. From this we can expect that a lot of waves are necessary for the system to regain stability. In large avalanches, a large amount of sand is transported. Part of this amount has to leave the system through the boundary sites, but since there are only three of these, we expect that a lot of waves will occur (as was already indicated by the exact result (3.33)). This is an important difference with the two dimensional case.
4.2 Stationary state properties

In this section we will analyse some stationary state properties of the BTW sandpile model on a Sierpinski gasket. By stationary state properties we mean properties which are not sufficient enough to describe the complex dynamics of the model, but remain constant or are smoothly fluctuating in the stationary SOC state. They are mere characterisations of the SOC state and cannot explain fully the dynamical aspects of the models.

For the BTW model a series of these stationary state properties can be calculated exactly, using the correspondence with the Potts model and exact properties of the Laplacian.

The mechanism applied here to the Sierpinski gasket both using the Potts model and the lattice Green function is quite general and can be extended to other lattices.

4.2.1 The number of recurrent configurations

The derivation of the number of recurrent configurations comes down to calculating the Potts partition function $\mathcal{Z}(K, q)$ in the $q \to 0$ limit. Then the number of spanning trees on a graph $G'$ - which consists of a Sierpinski gasket of $n$ generations with an extra sink site - is, following (2.22):

$$S(n) = \lim_{q \to 0} \frac{\mathcal{Z}(K, q)}{K^{v(G')-1}q^{v(G')+1}}.$$

Using $v(G') = v(n) + 1$ this becomes in terms of the function $v(n)$ defined in (4.2):

$$S(n) = \lim_{q \to 0} \frac{\mathcal{Z}(K, q)}{(\sqrt{qK})^{v(n)}q}.$$  \hspace{1cm} (4.7)

For a finitely ramified deterministic fractal like the Sierpinski gasket, the Potts partition function can be calculated exactly by a real space renormalisation procedure. This is what will be illustrated in this section.

Once the renormalisation group equations are found also critical exponents can be calculated.

1. Outline of the renormalisation

What we will calculate is the partition function $\mathcal{Z}(K, q)$ of the Potts model,
\[ Z(K, q) = \text{Tr} \ e^{\sqrt{q} K \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j}} \]  
(4.8)

where \( \langle ij \rangle \) stands for all couples of neighbouring sites. These couples can be graphically represented by bonds on a lattice. The principle of the renormalisation is to perform only the trace over the spins that appear only in the highest generation of the Sierpinski gasket. Performing the trace eliminates the spins in the highest generation, and then we proceed iteratively:

\[ Z(K, q) = \text{Tr}_{\{g=0\}} \text{Tr}_{\{g=1\}} \cdots \text{Tr}_{\{g=n-1\}} \text{Tr}_{\{g=n\}} e^{\sqrt{q} K \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j}} \]  
(4.9)

where \( g = m \) stands for the set of spins that appear only in the specific generation \( m \) of the gasket. We take the sink site to be in the \( g = 0 \) generation.

We split up the sum over all bonds in the exponent in a specific way. A Sierpinski gasket of generation \( n \) can be viewed as a collection of several elementary triangles (ET, see page 74), by which we mean triangles like a gasket of generation one, with six sites. Since an ET contains nine bonds, each ET will contribute nine terms in the sum in the exponential in (4.9). Hence the exponential in (4.9) can be written as a product over all ET in the gasket, times a product over all nine bonds in each specific ET,

\[ e^{\sqrt{q} K \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j}} = \prod_\alpha \prod_{\langle ij \rangle \in ET_\alpha} e^{\sqrt{q} K \delta_{\sigma_i, \sigma_j}}. \]  
(4.10)

To calculate \( Z \) we have to take the trace of (4.10) over all spins. But as clarified above we will first take only the trace over all spins that appear in the highest generation only. It is not hard to see that each ET has in fact a similar structure, with the three inner spins belonging to the highest generation of the gasket, and the border sites belonging to some lower generation. This enables us to perform the trace over the spins in the highest generation within a single ET and then derive from this the result for the entire gasket. For this we will look at one arbitrary ET and label the sites in it as presented in figure 4.3, namely: \( \{1, 2, 3\} \) for the corner sites, and \( \{4, 5, 6\} \) for the inner sites. Thus we will perform the trace of
\[ \prod_{\langle ij \rangle \in ET_\alpha} e^{\sqrt{q}K \delta_{\sigma_i, \sigma_j}} = \prod_{i=1, \ldots, 5 \atop j = i+1, \ldots, 6} e^{\sqrt{q}K \delta_{\sigma_i, \sigma_j}} \]  

(4.11)

only over the spins 4, 5 and 6, since they belong to the highest generation.

To calculate the trace we will use formula (2.14) again. Then the product over the nine bonds (4.11) becomes

\[ \prod_{i=1, \ldots, 5 \atop j = i+1, \ldots, 6} \left( 1 + \delta_{\sigma_i, \sigma_j} p \right) \]  

(4.12)

with \( p \equiv p(K, q) = e^{\sqrt{q}K} - 1 \) as in (2.15). Ultimately we are interested in the limit \( q \to 0 \), this corresponds to a limit \( p \to 0 \), and in this limit

\[ p^{q \to 0} \approx \sqrt{q}K. \]  

(4.13)

2. Constructing an iterative scheme

The result we get after taking the trace over the highest generation spins depends only on the spins of lower generations. To apply the idea of renormalisation, we should write this result in a comparable form to the original expression for \( Z \) (4.8). Doing this, it will be necessary to define new effective Hamiltonian parameters. The relation between these new parameters and the original ones will allow us to calculate critical exponents of the Potts model. We will come to this later.

When we want to write the result in a form similar to the original form after performing the trace of (4.11) over the inner spins, we will meet some difficulties. As usual in (real space) renormalisation calculations, there will be a proliferation of interactions. In particular, for the present an interaction of the form

\[ \delta_{\sigma_i, \sigma_j} \delta_{\sigma_j, \sigma_k} \]

will be generated, which we will denote by

\[ \delta_{\sigma_i, \sigma_j} \delta_{\sigma_j, \sigma_k} \equiv \delta_\Delta. \]  

(4.14)
To perform a correct renormalisation group procedure we will have to take into account this kind of interaction from the start. It is easy to see that taking into account this $\delta_\Delta$ interaction is not in contradiction with the fact that the result of the calculation should finally yield the number of spanning trees. A $\delta_\Delta$ interaction in the lowest level of the renormalisation (with the highest generation spins) can not generate a spanning tree, since it represents a loop. However an interaction term with e.g. five $\delta_{\sigma_i,\sigma_j}$ interactions like the example in figure 4.4 will generate a $\delta_\Delta$-interaction in the next renormalisation group step. However it can perfectly generate a spanning tree since there is no loop at the lowest level.

So the Hamiltonian of the generalised Potts model becomes

$$-\mathcal{H}(\vec{K}, q) = \sqrt{q}K \sum_{(ij)} \delta_{\sigma_i,\sigma_j} + q^\eta L \sum_\Delta \delta_\Delta \quad (4.15)$$

with $\vec{K} = (K, L)$. At this point it is not clear what $\eta$ should be. However since the $\delta_\Delta$-interaction is not in the general Potts Hamiltonian but is generated through the renormalisation process only, we expect it to vanish in the final results. It should certainly not play a role in the calculation of the number of spanning trees, since this $\delta_\Delta$ type of interaction represents a loop in the highest generation gasket, as explained above. There seems no specific way to fix $\eta$ with only the condition that it should not appear in the final results. Quite reasonable seems to
choose \( \eta \) such that the \( \delta_\Delta \) interaction term is of the same order as the lowest order interaction term of this kind when it is produced by the \( \delta_{\sigma_i,\sigma_j} \) interaction. This turns out to be 1 (see appendix A.1). Therefore we choose \( \eta = 1 \).

The sum over \( \Delta \) runs over all up\( \text{right} \) triangles in the system. Above we described that we should incorporate an extra triangular interaction because the renormalisation procedure produces it. It is clear that the renormalisation can only produce upright triangles, and never a triangular interaction of the type \( \delta_{\sigma_4,\sigma_5}\delta_{\sigma_5,\sigma_6} \). Therefore we do not have to take it into account when starting the renormalisation group procedure.

The partition function becomes

\[
\mathcal{Z}(K, q) = \text{Tr} e^{\sqrt{q}K \sum_{(ij)} \delta_{\sigma_i,\sigma_j} + qL \sum_{\Delta} \delta_\Delta} \quad (4.16)
\]

\[
= \text{Tr} \prod_\alpha \prod_{(ij), \Delta \in ET_\alpha} e^{\sqrt{q}K \delta_{\sigma_i,\sigma_j} + qL \delta_\Delta}. \quad (4.17)
\]

Denoting

\[
\bar{p} = \bar{p}(L, q) = e^{qL} - 1
\]

the expression analogous to (4.12) for an arbitrary \( ET_\alpha \) becomes:

\[
\prod_{(i,j) \in ET_\alpha} \left(1 + \delta_{\sigma_i,\sigma_j} \bar{p}\right) \prod_{k=1,\ldots,3} \left(1 + \delta_\Delta \bar{p}\right). \quad (4.18)
\]

It is this expression over which we have to take the trace, over the spins \( \{\sigma_4,\sigma_5,\sigma_6\} \). In (4.18) we use the numbering \( \{\Delta_k, k = 1, 2, 3\} \) of upright triangles indicated in figure 4.5.

To set up an iterative scheme we need to write the Hamiltonian on the next renormalisation level in the same form as the original Hamiltonian (4.15), but with renormalised parameters \( \bar{K}' = (K', L') \). Moreover, as usual [30] a constant term will be generated. This represents the free energy of the eliminated spins. \( \mathcal{Z} \) then becomes

\[
\mathcal{Z}(\bar{K}, q) = \text{Tr}_{(g=0)} \ldots \text{Tr}_{(g=n-1)} e^{G^{(n)} + \sqrt{q}K' \sum_{(ij)} \delta_{\sigma_i,\sigma_j} + qL' \sum_{\Delta} \delta_\Delta} \quad (4.19)
\]

\[
= \text{Tr}_{(g=0)} \ldots \text{Tr}_{(g=n-1)} e^{G^{(n)} + \sqrt{q}K' \sum_{(ij)} \delta_{\sigma_i,\sigma_j} + qL' \sum_{\Delta} \delta_\Delta} \quad (4.20)
\]
with $G^{(n)} \equiv G^{(n)}(\vec{K}, q)$ the contribution to the free energy of the eliminated spins, and $K' = K'(\vec{K})$ and $L' = L'(\vec{K})$.

The expression analogous to (4.18) for the triangle between $\sigma_1, \sigma_2$ and $\sigma_3$ in the lattice of generation $n - 1$ is

$$\prod_{i=1,2} \prod_{j=i+1,3} (1 + \delta_{\sigma_i, \sigma_j} p') (1 + \delta_{\Delta} \tilde{p}')$$

(4.21)

where $p' = \sqrt{q}K'$ and $\tilde{p}' = qL'$. Working out (4.21) we find up to the lowest order in $q$ for each kind of interaction term:

$$1 + \sqrt{q}K' \sum_{i,j=1 \atop i \neq j}^{3} \delta_{\sigma_i, \sigma_j} + q \left(3K'^2 + L' \right) \delta_{\Delta}.$$  

(4.22)

It suffices to take into account only the lowest order contributions in $q$ in each term because we are interested only in the $q \to 0$ limit of the partition function. It is formula (4.22) that we should compare with the result of the trace from (4.18).

3. Derivation of the renormalisation group equations

The calculation of the trace from (4.18) over $\sigma_4, \sigma_5$ and $\sigma_6$ is extensive but straightforward. It is presented in appendix A.2. The result up to the lowest order in $q$ for each interaction term is
\[ a(\vec{K})q^{\frac{3}{2}} + b(\vec{K})q^2 \sum_{i,j=1 \atop i \neq j}^3 \delta_{\sigma_i,\sigma_j} + c(\vec{K})q^{\frac{5}{2}} \delta_\Delta \quad (4.23) \]

where

\[
\begin{align*}
  a(\vec{K}) &= 50K^3 + 12KL \\
  b(\vec{K}) &= 30K^4 + 13K^2L + L^2 \\
  c(\vec{K}) &= 54K^5 + 36K^3L + 6KL^2.
\end{align*}
\quad (4.24)
\]

Comparing this with (4.22) we see that we need to write (4.23) as

\[ a(\vec{K})q^{\frac{3}{2}} \left( 1 + \sqrt{q} \bar{b}(\vec{K}) \sum_{i,j=1,\ldots,3} \delta_{\sigma_i,\sigma_j} + q\bar{c}(\vec{K})\delta_\Delta \right) \quad (4.25) \]

with \( \bar{b}(\vec{K}) = b(\vec{K}) / a(\vec{K}) \) and \( \bar{c}(\vec{K}) = c(\vec{K}) / a(\vec{K}) \). The factor \( a(\vec{K})q^{\frac{3}{2}} \) is the same in all ET of the \( n \)th generation, and when we bring it in front of the product over all ET of the \( n \)th generation in (4.17), we can identify it with the factor \( e^{G(n)} \) in (4.20):

\[ e^{G(n)} = \left( a(\vec{K})q^{\frac{3}{2}} \right)^{N^{(n)}_{ET}} \quad (4.26) \]

where \( N^{(n)}_{ET} \) is the number of ET in the gasket of generation \( n \), \( N^{(n)}_{ET} = 3^{n-1} \).

Comparing (4.22) and (4.23) we can derive the renormalisation group equations, since the product over all elementary triangles in the gasket of generation \( n \) corresponds to the product over all triangles in the gasket of generation \( n - 1 \). We find

\[
\begin{align*}
  K' &= \bar{b} \\
  L' &= \bar{c} - 3\bar{b}^2
\end{align*}
\quad (4.27)
\]

or

\[
\begin{align*}
  K' &= \frac{30K^4 + 13K^2L + L^2}{50K^3 + 12KL} \\
  L' &= \frac{54K^5 + 36K^3L + 6KL^2}{50K^3 + 12KL} - 3 \left( \frac{30K^4 + 13K^2L + L^2}{50K^3 + 12KL} \right)^2.
\end{align*}
\quad (4.28)\]
These are the renormalisation equations of the Potts model on the Sierpinski gasket in the limit \( q \to 0 \).

4. **The critical point of the \( q \to 0 \) Potts model**

To find a critical point we need to find the fixed point of the equations (4.28). Indeed, at the critical point the system exhibits scale invariance and hence the renormalisation should not alter the Hamiltonian parameters. The only solution is the fixed point \( \vec{K}^\ast = (0, 0) \). \( L = 0 \) corresponds to our expectation since the limit \( q \to 0 \) is known to yield the spanning trees of the lattice, hence \( L^\ast \) surely has to be 0 (as explained above).

The renormalisation matrix becomes (putting \( L = 0 \) but still leaving \( K \) indefinite)

\[
\mathcal{R} = \begin{pmatrix}
\frac{3}{5} & \frac{29}{250} & 1 \\
\frac{3}{5} & \frac{1}{250} & \frac{1}{10} \\
0 & \frac{27}{625} & \frac{1}{5}
\end{pmatrix}.
\]

(4.29)

It has the eigenvalues (which are independent of \( K \))

\[
\lambda_1 = \frac{3}{5}, \quad \lambda_2 = \frac{27}{625} = \frac{1}{5} \left( \frac{3}{5} \right)^3.
\]

The critical exponents \( y_t \) and \( y_L \) are defined by

\[
\lambda_1 = b^{y_t}, \quad \lambda_2 = b^{y_L}.
\]

with \( b \) the fundamental rescaling factor of the Sierpinski gasket, \( b = 2 \). We find

\[
y_t = \frac{\log 3/5}{\log 2} = D_f - D_{rw} = -0.73696\ldots
\]

\[
y_L = \frac{\log 28/625}{\log 2} = 3D_f - 4D_{rw} = -4.53282\ldots
\]

The exponent \( x_t \) that describes the decay of the Potts correlation function is related to \( y_t \) through
4.2. **STATIONARY STATE PROPERTIES**

\[ x_t + y_t = D_f. \]  \hfill (4.30)

The result is

\[ x_t = \frac{\log \frac{5}{2}}{\log 2} = D_{rw} = 2.32192 \ldots \]  \hfill (4.31)

We see that the exponent \( x_t \) corresponds to the walk dimension of the Sierpinski gasket while minus the exponent \( y_t \) describes the decay of the lattice Green function and hence also of the expected number of waves in an avalanche. This clearly relates exponents in the Potts model and the BTW sandpile model.

5. **The partition function of the \( q \to 0 \) Potts model**

Since the factor (4.26) is spin-independent we can put it in front of the whole trace in (4.20). Each trace over the spins in a specific generation yields such a contribution. This renormalisation procedure therefore allows us to write the partition function \( Z^{(n)} \) of the Potts model on the Sierpinski gasket of \( n \) generations iteratively as

\[
Z^{(n)}(\vec{K}, q) = [a(\vec{K}^{(0)})q^{\frac{3}{2}}]^{3^n-1} Z^{(n-1)}(\vec{K}^{(1)}, q) \\
= \left[a(\vec{K}^{(0)})q^{\frac{3}{2}}\right]^{3^{n-1}} \left[a(\vec{K}^{(1)})q^{\frac{3}{2}}\right]^{3^{n-2}} Z^{(n-2)}(\vec{K}^{(2)}, q) \\
= \ldots \\
= \left[a(\vec{K}^{(0)})q^{\frac{3}{2}}\right]^{3^{n-1}} \left[a(\vec{K}^{(1)})q^{\frac{3}{2}}\right]^{3^{n-2}} \ldots \\
\ldots \left[a(\vec{K}^{(n-1)})q^{\frac{3}{2}}\right]^{3^0} Z^{(0)}(\vec{K}^{(n)}, q) \\
= \prod_{i=0}^{n-1} \left[a(\vec{K}^{(n-i)})\right]^{3^i} q^{\frac{3}{2}} \sum_{i=0}^{n-1} 3^i Z^{(0)}(\vec{K}^{(n)}, q) \]  \hfill (4.32)

with \( \vec{K}^{(0)} = \vec{K} \), and \( \vec{K}^{(m)} = \vec{K}^{(m)} \left(\vec{K}^{(m-1)}\right) \) given by the renormalisation equations (4.28).

Performing the sum in the exponent of \( q \), (4.32) becomes

\[
Z^{(n)}(\vec{K}, q) = \prod_{i=1}^{n} \left[a(\vec{K}^{(n-i)})\right]^{3^{i-1}} q^{\frac{3}{4}(3^n-1)} Z^{(0)}(\vec{K}^{(n)}, q). \]  \hfill (4.33)
So finally to calculate the number of spanning trees we have to calculate $Z^{(n)}((K, L = 0), q)$. Because (4.28) never generates an $L$ during the renormalisation when we start from $L = 0$, we can neglect the flow for $L$ to e.g. calculate $a(\tilde{K}^{(j)})$. Using the flow for $K$ (4.28) and the definition of $a(\tilde{K})$ (4.24):

$$a(\tilde{K}^{(j)}) = 50K^3 \left( \frac{3}{5} \right)^{3j}.$$ 

The factor with the product of $a$ in (4.33) can be divided in two parts. The first one is

$$\prod_{i=1}^{n} \left(50K^3\right)^{3i-1} = \left(50K^3\right) \sum_{i=1}^{n} 3^{i-1} = \left(50K^3\right) \frac{3^{n-1}}{2}. \quad (4.34)$$

The second one is

$$\prod_{i=1}^{n} \frac{3}{5}^{(n-i)3i} = \left(\frac{3}{5}\right)^{\frac{3}{2}(3^{n-1})-\frac{3}{2}n}. \quad (4.34)$$

We will use formula (4.2) for the number of sites in a Sierpinski gasket of generation $n$. Also we will denote

$$\mu = \sqrt[3]{50} \sqrt{\frac{3}{5}} = 2.85363 \ldots \quad (4.35)$$

Then (4.33) becomes

$$Z^{(n)}(\tilde{K}, q) = \frac{\mu^{v(n)} \left(\sqrt{qK}\right)^{v(n)}}{50K^3q^{\frac{3}{2}} \left(\frac{3}{5}\right)^{\frac{3}{2}(n+1)}} Z^{(0)}(\tilde{K}^{(n)}, q). \quad (4.36)$$

The trace $Z^{(0)}(\tilde{K}^{(n)}, q)$ over the spins in the lowest generation is not so hard to calculate directly. The calculation is done in appendix A.3. The lowest order term in $q$ is found to be

$$Z^{(0)}(\tilde{K}^{(n)}, q) = \left(8K^3 + 24K^2 K^{(n)} + 18K \left(K^{(n)}\right)^2 \right) q^\frac{5}{2} \quad (4.37)$$

$$= \left(8 + 24 \left(\frac{3}{5}\right)^n + 18 \left(\frac{3}{5}\right)^{2n} \right) q^\frac{5}{2} K^3. \quad (4.38)$$
Finally we have the exact expression for the partition function of the $q \to 0$ Potts model on a Sierpinski gasket of $n$ generations:

$$Z^{(n)}(K,q) = \mu^{v(n)} \left( \sqrt{q} K \right)^{v(n)} q f(n)$$  \hspace{1cm} (4.39)

with

$$f(n) = \frac{8 + 24 \left( \frac{3}{5} \right)^n + 18 \left( \frac{3}{5} \right)^{2n}}{50 \left( \frac{3}{5} \right)^{(2n+1)}/5)}.$$  \hspace{1cm} (4.40)

We use this result together with (4.7) and finally we find the number of spanning trees on a Sierpinski gasket of $n$ generations with an extra sink site:

$$S(n) = \mu^{v(n)} f(n).$$  \hspace{1cm} (4.41)

This is an exact result. $S(n)$ also equals the number of recurrent configurations of the BTW model on a Sierpinski gasket of $n$ generations. The quantity $\mu$ is called the connective constant for spanning trees.

We can rewrite (4.41) in a form more convenient in statistical mechanics, namely

$$S(n) \sim \mu^{v(n)} v(n)^{-\theta}$$  \hspace{1cm} (4.42)

with $\theta$ some exponent. Since in the limit for large $n$

$$f(n) \overset{n \to \infty}{\sim} \left( \frac{5}{3} \right)^{(3/2)n},$$

we find

$$\theta = \frac{3 \log 5/3}{2 \log 3}.$$  \hspace{1cm} (4.43)

As $n$ grows (4.42) becomes

$$\frac{\log S(n)}{v(n)} \overset{n \to \infty}{\sim} \log \mu.$$  \hspace{1cm} (4.44)
<table>
<thead>
<tr>
<th>$n$</th>
<th>$\log S(n)/v(n)$</th>
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<tr>
<td>0</td>
<td>3.56087...</td>
</tr>
<tr>
<td>1</td>
<td>3.36197...</td>
</tr>
<tr>
<td>2</td>
<td>3.11815...</td>
</tr>
<tr>
<td>3</td>
<td>2.97814...</td>
</tr>
<tr>
<td>4</td>
<td>2.90877...</td>
</tr>
<tr>
<td>5</td>
<td>2.87701...</td>
</tr>
<tr>
<td>6</td>
<td>2.86320...</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\mu = 2.85363...$</td>
</tr>
</tbody>
</table>

Table 4.1: Evolution of $\log S(n)/v(n)$ in function of the gasket generation $n$ using (4.41). The limit is the connective constant $\mu$.

**Numerical verification**

The connective constant can also be calculated numerically by calculating the determinant of the toppling matrix $\Delta$. For the gaskets of generation $n \leq 3$ this can be done by a program like *Mathematica*®. For higher generation gaskets this can be calculated by a numerical procedure in e.g. Fortran. Since it is also of interest to calculate the inverse of $\Delta$ (see section 4.2.2) a row equivalence scheme for inverting matrices is interesting because such a routine requires the storage of diagonal elements of the triangular form of $\Delta$, and the determinant of course is simply the product of these. We numerically verified that indeed $S(n)$ from (4.41) and $\text{det}(\Delta)$ coincide. The evolution of $\log S(n)/v(n)$ is presented in table 4.1.

**4.2.2 Properties derived from the lattice Green function**

The lattice Green function related to the discrete Laplacian gives us interesting information on stationary state properties of the BTW model. In the first place Dhar argued (section 2.2.5 and [27]) and Ivashkevich, Ktitarev and Priezzhev proved (section 3.2.2 and [39]) that $G_{ij}$ stands for the expected number of topplings at site $j$ when an avalanche was initiated at site $i$. Then $G_{ii}$ stands for the expected number of waves in an avalanche initiated at site $i$. Also e.g. the stationary state property $P(1)$, i.e. the expectation value that a site has height equal to 1 in the SOC state, can be calculated following the scheme developped by Majumdar and Dhar [26], which was extended
what concerns the notation for a general graph in section 3.1.2.

- **The lattice Green function on the Sierpinski gasket**

  The lattice Green function $G$ on the Sierpinski gasket can be numerically determined by simply inverting the toppling matrix $\Delta$ (1.1). Of course, it then depends on the practical computing capacity up till which generation $G$ can be calculated. A program like *Mathematica* should have no problems up to the third generation gasket (with $v(3) = 42$ sites and hence $42 \times 42$ matrices). As an example, with the numbering of sites as indicated in figure 4.3, the Green function for the first generation gasket can be found by inversion of (4.6), giving

  $$G = \frac{1}{38} \begin{pmatrix}
  13 & 3 & 3 & 7 & 5 & 7 \\
  3 & 13 & 3 & 7 & 7 & 5 \\
  3 & 3 & 13 & 5 & 7 & 7 \\
  7 & 7 & 5 & 18 & 10 & 10 \\
  5 & 7 & 7 & 10 & 18 & 10 \\
  7 & 5 & 7 & 10 & 10 & 18 
\end{pmatrix}. \quad (4.45)$$

  For higher generation gaskets the Green function can be calculated by a numerical procedure in e.g. Fortran. The simple row equivalence scheme for inverting matrices, that can be found in any first grade algebra handbook, was found to be very accurate and fast up to generation $n = 10$. It has the advantage that the determinant of $\Delta$ can be found simultaneously because the inversion routine needs the storage of diagonal elements of the triangular form of $\Delta$.

- **The expected number of sites with height 1**

  Using the scheme developed in section 3.1.2 (based on Majumdar and Dhar’s work [26] on the two dimensional square lattice) we can calculate $P(1)$, i.e. the expected number of sites with height 1 in the stationary state for the BTW model on the Sierpinski gasket.

  The quantity $P_1(1)$ in (3.6) is site dependent on the Sierpinski gasket. Note that, due to the symmetries of the Sierpinski gasket, we can distinguish several classes of sites which have a comparable environment. Within each class all sites assume the same value for $P_1(1)$. Then (3.5) can be replaced by a weighted sum over all these classes. This is practical for low generation gaskets. For higher generation gaskets it is more practical to calculate $P_1(1)$ for each site in the gasket and use (3.5). As an example, the gasket of generation $n = 1$ has 2 such classes, that of
Table 4.2: Exact results for the fraction $P(1)$ of sites having height equal to one in the BTW model on Sierpinski gaskets of generation $n$, compared to numerical results from [58].

<table>
<thead>
<tr>
<th>$n$</th>
<th>$P(1)$ exact</th>
<th>$P(1)$ from [58]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.115650...</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>0.087606...</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>0.075055...</td>
<td>0.075</td>
</tr>
<tr>
<td>4</td>
<td>0.070277...</td>
<td>0.070</td>
</tr>
<tr>
<td>5</td>
<td>0.068580...</td>
<td>0.069</td>
</tr>
<tr>
<td>6</td>
<td>0.067995...</td>
<td>0.068</td>
</tr>
<tr>
<td>7</td>
<td>0.067797...</td>
<td>0.068</td>
</tr>
</tbody>
</table>

generation $n = 2$ has 4 such classes and the gasket of generation $n = 3$ has 9 such classes of sites.

The results for $P(1)$ for gaskets of different generations are presented in table 4.2. Note that these values are numerically determined but are nevertheless exact. They are in coincidence with the numerical results of [58] obtained from simulations.

It is interesting to look at the behaviour of $P_1(1)$ on the gasket. To keep the presentation of data simple, let us look at the behaviour of $P_1(1)$ along an edge of the gasket. The data for the gasket of generation $n = 7$ are presented in figure 4.6. $P_1(1)$ turns out to be a wildly fluctuating function. Perhaps it might be a self-affine curve but this was not examined further. When we investigate the behaviour of $P_1(1)$ along an edge with increasing gasket generation, it turns out that $P_1(1)$ does not become a smooth function but instead becomes more and more sharply peaked. To illustrate this we present in figure 4.7 the exact data for $P_1(1)$ for the qualitatively comparable sites in different generations. E.g. the corner sites are qualitatively comparable in all generations, as is the central site along an edge. The second and fourth site along an edge, starting counting at a corner site, in the gasket of generation 2 are qualitatively comparable to the third and seventh site in the gasket of generation 3, and so on.
4.2. **STATIONARY STATE PROPERTIES**

Figure 4.6: $P_i(1)$ along an edge of the Sierpinski gasket of generation $n = 7$.  

Figure 4.7: $P_i(1)$ for qualitatively comparable sites along an edge of the Sierpinski gasket for generations $n = 3$ (bullets), $4$ (diamonds) and $5$ (triangles).
• The expected number of topplings

Using (2.47) and exact values for the lattice Green function on the Sierpinski gasket we can exactly calculate the expected number of topplings \( \langle s \rangle \) in the BTW model on a Sierpinski gasket. These data are presented in table 4.3. The evolution of

\[
\frac{\log (\langle s \rangle_{L_{n+1}}/\langle s \rangle_{L_n})}{\log (L_{n+1}/L_n)},
\]

with \( L_n \) and \( L_{n+1} \) the linear sizes of gaskets of successive generations, is in correspondence with relation (3.31).

• The expected number of waves

Using (3.32) and exact values for the lattice Green function on the Sierpinski gasket we can exactly calculate the expected number of waves \( \langle n_w \rangle \) in the BTW model on a Sierpinski gasket. These data are presented in table 4.4. The evolution of

\[
\frac{\log (\langle n_w \rangle_{L_{n+1}}/\langle n_w \rangle_{L_n})}{\log (L_{n+1}/L_n)}
\]

is in correspondence with relation (3.33).

### 4.3 Dynamical aspects and scaling

In this section we will present results on the dynamics of the BTW model on the Sierpinski gasket. We will finally examine the scaling behaviour of this model. As can be concluded from the previous chapters, little is known exactly on the dynamical aspects of the BTW model. Most of the results in this section will come from computer simulations.

By *dynamical aspects* we mean the wave and avalanche statistics, correlations between waves (both on short and long time scales), and the scaling behaviour of various functions. We will start by the basic constituents - waves - and will follow the way up towards avalanches.
4.3. DYNAMICAL ASPECTS AND SCALING

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \langle s \rangle )</th>
<th>( \log \left( \frac{\langle s \rangle_{L_{n+1}}}{\langle s \rangle_{L_n}} \right) / \log \left( \frac{L_{n+1}}{L_n} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{5}{4} = 1.25 )</td>
<td>1.7824…</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{43}{10} = 4.3 )</td>
<td>1.9801…</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{475}{28} = 16.96428… )</td>
<td>2.0828…</td>
</tr>
<tr>
<td>4</td>
<td>71.86585…</td>
<td>2.1559…</td>
</tr>
<tr>
<td>5</td>
<td>320.26639…</td>
<td>2.2109…</td>
</tr>
<tr>
<td>6</td>
<td>1482.71643…</td>
<td>2.2567…</td>
</tr>
<tr>
<td>7</td>
<td>7085.8…</td>
<td>-</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( D_{rw} = 2.3219… )</td>
</tr>
</tbody>
</table>

Table 4.3: Exact results for the expected number of topplings in an avalanche \( \langle s \rangle \) for the BTW model on Sierpinski gaskets of various generations. The evolution of the quantity in the right column corresponds with (3.31).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \langle n_w \rangle )</th>
<th>( \log \left( \frac{\langle n_w \rangle_{L_{n+1}}}{\langle n_w \rangle_{L_n}} \right) / \log \left( \frac{L_{n+1}}{L_n} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{31}{76} = 0.40789… )</td>
<td>0.5731…</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{7009}{11550} = 0.60683… )</td>
<td>0.6204…</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{43231}{46340} = 0.93290… )</td>
<td>0.6474…</td>
</tr>
<tr>
<td>4</td>
<td>1.46133…</td>
<td>0.6719…</td>
</tr>
<tr>
<td>5</td>
<td>2.32823…</td>
<td>0.6926…</td>
</tr>
<tr>
<td>6</td>
<td>3.76305…</td>
<td>0.7077…</td>
</tr>
<tr>
<td>7</td>
<td>6.14603…</td>
<td>-</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( D_{rw} - D_f = 0.73696… )</td>
</tr>
</tbody>
</table>

Table 4.4: Exact results for the expected number of waves within an avalanche \( \langle n_w \rangle \) for the BTW model on Sierpinski gaskets of various generations. The evolution of the quantity in the right column corresponds with (3.33).
4.3.1 Computer simulations on the Sierpinski gasket

In order to perform computer simulations of sandpile models on the Sierpinski gasket a code in Fortran 77 was written. The algorithm used is rather straightforward. First on a square lattice of appropriate size a Sierpinski gasket was constructed by shifting small triangles over the lattice. Each site in the gasket was given a code indicating the directions in which to find its nearest neighbours. Then the sites of the Sierpinski gasket were labelled and the listed together with the labels of their neighbours in an $4 \times N$ matrix.

The sandpile models were simulated on an $k \times N$ matrix, the first row containing the real system information $(z_i)$. The number $k$ can be minimised by appropriate encoding. When a site topples the $N \times 4$ table was used to look up the neighbours of this site. Rather than copying the entire system, unstable sites were put in a special list. The $k - 1$ other rows of the system matrix were used to indicate whether a site was already in this special list, whether it toppled already in the wave or avalanche, and to keep track of its appropriate toppling time in the avalanche (following the definition on page 30). After each wave these rows were used to calculate the wave area, and after each avalanche to calculate the avalanche area and duration.

Unless otherwise indicated the data presented here are collected from simulations of the BTW model on Sierpinski gaskets of generations 5 up till 9. On each of them $10^7$ avalanches were ignited on randomly selected sites and separated in time, as the models prescribe. These simulation parameters are mainly determined by limitations in computing and storing capacity.

4.3.2 Statistics of individual waves

The probability distribution function for waves in gaskets of various generations, as determined by the computer simulations, is presented in figure 4.8. It is calculated by interpreting the sandpile evolution as a consecutive series of waves, and disregarding to which avalanche the waves belong. In the next figure, figure 4.9, the probability distribution function for the last wave in each avalanche is presented. It is clear that the larger the gasket, the larger the maximum possible wave size is, and this rule determines which probability distribution function corresponds to which gasket.

A first striking feature in these figures is the presence of log-periodic oscillations due to the fractal geometry of the underlying lattice as was explained in section 2.3.2. Besides the log-periodic oscillations various irregularities can be observed in the functions (they are not smooth).
Figure 4.8: Probability distribution function of wave sizes in the BTW model on the Sierpinski gasket.

Figure 4.9: Probability distribution function of last wave sizes in the BTW model on the Sierpinski gasket.
To examine the scaling behaviour one could try a plot of e.g. \( w^{\tau_w} P(w) \) versus \( w/L_w^D \) and see if the probability distribution functions for different system sizes collapse. That would be the case for finite size scaling. Our experience is that this usually cannot be done so easily for these probability distribution functions on the Sierpinski gasket. The irregularities considerably complicate this task. Therefore we use the moment and multifractal analyses that were presented in section 2.3.

In figure 4.10 we present the moment scaling functions \( \sigma_w(q) \) and \( \sigma_t(q) \) following the definition of (2.70) for waves and last waves respectively, computed from the probability distribution functions of figures 4.8 and 4.9 respectively. As in most similar computations in this thesis we choose an interval of 0.1 in \( q \)-values for \( q \in [-1, 1] \) and for an interval of 1 in \( q \)-values elsewhere, and we limit \( q \) to the interval \( q \in [-10, 10] \). This different resolution for \( q \) near 0 allows for a detailed analysis in this important region.

It is striking that no trace can be seen of the irregularities and oscillations of figures 4.8 and 4.9. Disregarding a continuous bending of the functions for \( q \approx 0 \) that is due to the finite system sizes [33] and logarithmic corrections (see page 38) the functions have a behaviour similar to the behaviour of (2.73), or to figure 2.3 (for \( \sigma_w \)) and to figure 2.2 (for \( \sigma_t \)). This is a strong indication that wave size distribution as well as last wave size distribution functions obey finite size scaling. The fact that both \( \sigma_w(q) \) and \( \sigma_t(q) \) become linear functions for \( q > 0 \) after a small transition region can be seen from figure 4.11 in which the local slope \( \Delta \sigma/\Delta q \) of these functions is presented.

From figure 4.11 we find

\[
D_w = 1.58 \pm 0.01
\]

\[
D_t = 1.58 \pm 0.01
\]  

which coincides with the fractal dimension \( D_f \) (4.1) of the Sierpinski gasket. This is quite logical, since in a wave (and hence also last wave) all sites can topple only once and hence the wave size is limited by the number of sites (4.3).

From figure 4.10 we find

\[
\tau_w = 0.56 \pm 0.03
\]

\[
\tau_t = 1.25 \pm 0.03
\]

These numerical results can now be compared with the exact results (3.20) and (3.26). Using (4.1), (4.4) and (4.5), we find as exact values on the Sierpinski gasket:
4.3. DYNAMICAL ASPECTS AND SCALING

Figure 4.10: Moment scaling functions $\sigma_w(q)$ (squares) and $\sigma_l(q)$ (diamonds) for waves and last waves in the BTW model on the Sierpinski gasket, computed from the data of figures 4.8 and 4.9.

Figure 4.11: Local slope $\Delta \sigma / \Delta q$ of the moment scaling functions $\sigma_w(q)$ (squares) and $\sigma_l(q)$ (diamonds) for waves and last waves in the BTW model on the Sierpinski gasket, computed from the data of figure 4.10.
\[
\tau_w = \frac{\log 9/5}{\log 3} = 0.535 \ldots \quad (4.50)
\]
\[
\tau_l = \frac{\log 135/(20 + \sqrt{205})}{\log 3} = 1.247 \ldots \quad (4.51)
\]

and we can conclude that there is coincidence between the numerical obtained data and the predictions from section 3.2.

There is no indication that a relation as (2.55) should hold between general wave and last wave sizes. Indeed relation (2.55) is expected to hold between geometrical and dynamical properties of one kind of entity (waves, last waves, avalanches, \ldots) but not necessarily between properties of different entities. Hence we cannot expect to find an invariant as (2.57). However we can, inspired by (2.57), define ad hoc

\[
\zeta_w \equiv D_w(\tau_w - 1)
\]
\[
\zeta_l \equiv D_l(\tau_l - 1).
\]

\(\zeta_w\) and \(\zeta_l\) indicate important marking points in the moment scaling function and multifractal spectrum plots (see figures 2.2 and 2.3). The exact values for \(\zeta_w\) and \(\zeta_l\) are

\[
\zeta_w = \frac{\log 3/5}{\log 2} = -0.736 \ldots \quad (4.52)
\]
\[
\zeta_l = \frac{\log 45/(20 + \sqrt{205})}{\log 2} = 0.390 \ldots . \quad (4.53)
\]

From the numerical data (4.46)-(4.47) and (4.48)-(4.49) we find

\[
\zeta_w = -0.69 \pm 0.05 \quad (4.54)
\]
\[
\zeta_l = 0.41 \pm 0.05. \quad (4.55)
\]

Now we can look at the multifractal spectrum obtained from the numerical data. In figure 4.12 the finite system size spectra are presented together with an extrapolated \(L \rightarrow \infty\) spectrum. The finite system size spectra (for gaskets of generations 5 up till 9) are obtained using the parametrised spectra expressions (2.65)-(2.66). The extrapolation method used here comes from [33]. The finite system size values \(\alpha_L\) and \(f_L(\alpha)\) are used to plot respectively \(\alpha_L \log L\) and \(f_L(\alpha) \log L\) versus \(\log L\). As \(L\) grows, these plots should
Figure 4.12: Finite size multifractal spectrum for the wave size distribution in the BTW model on the Sierpinski gasket, computed from the data of figure 4.8. The triangles represent an extrapolated curve for $L \to \infty$.

Figure 4.13: Finite size multifractal spectrum for the last wave size distribution in the BTW model on the Sierpinski gasket, computed from the data of figure 4.8. The triangles represent an extrapolated curve for $L \to \infty$. 
Figure 4.14: Extrapolated multifractal spectra for the wave and last wave size distribution in the BTW model on the Sierpinski gasket. Because the data from which is extrapolated are from small systems, the extrapolated forms are still rather far from the ideal $L \to \infty$ forms which are simple piecewise linear curves.

become linear, and the slope gives the extrapolated value of $\alpha_L$ and $f_L(\alpha)$ respectively. As explained in [33] and already mentioned on page 36 here, even the extrapolated curve can be far from the exact one. However in [33] is was extensively shown that one certitude indicating finite size scaling is an accumulation of points in the extrapolated form around $(D_x, -\zeta)$.

From the moment analysis we have a strong indication of finite size scaling for the wave probability distribution function. The accumulation of points in the extrapolated curve in figure 4.12 around $(D_w, -\zeta_w)$ can be a confirmation of this. The extrapolated curve is still far from the linear form, but this is mainly due to the small system sizes used here and the limited number of avalanches.

In the same way a multifractal analysis can be done of the last wave size probability distribution function. The finite system size spectra are presented in figure 4.13 together with the extrapolated spectrum. Again an accumulation of points is observed, now around the point $(D_l, -\zeta_l)$. This can again confirm the finite size scaling behaviour that was indicated already by the moment analysis.
Figure 4.15: Collapse plot for the wave size probability distribution function of figure 4.8 using the exponent values (4.46)-(4.47) and (4.48)-(4.49) that were obtained by moment analysis.

Figure 4.16: Collapse plot for the last wave size probability distribution function of figure 4.9 using the exponent values (4.46)-(4.47) and (4.48)-(4.49) that were obtained by moment analysis.
In figure 4.14 we present only the extrapolated spectra for waves and last waves, together with the ultimate piecewise linear $L \to \infty$ form.

Finally one may wonder how the 'classical' collapse plots may look like for these probability distribution functions. Once the data (4.46)-(4.47) and (4.48)-(4.49) are known we can use them to make collapse plots for the wave and last wave probability distribution functions. The results are presented in figures 4.15 and 4.16 and these show indeed a good collapse. Let it be clear that trying to collapse these plots without knowing (4.46)-(4.47) and (4.48)-(4.49), and only by trial and error, usually is a tedious work in which one has few certainties. Different parts of different plots collapse differently, and the question is what priority should one give to which part. In fact results obtained before with the collapse method [59] are rather different from the results obtained by the moment analysis.

The moment analysis is much more transparant, works very good in estimating the scaling behaviour, and in the case of finite size scaling it yields good values for the exponents. Therefore we recommend it, in combination with the multifractal analysis, as the most reliable method for determining scaling behaviour in various models like sandpile models on fractal lattices, on which probability distribution functions mostly take on rather irregular forms.

We can conclude by saying that we have found strong indications that the general wave and last wave probability distribution functions obey finite size scaling, with exponents given by (4.46)-(4.47) and (4.48)-(4.49). The results from numerical investigation of their respective asymptotic behaviour are in good correspondence with the theoretical predicted forms (3.20) and (3.26).

### 4.3.3 Statistics of consecutive waves

We will now examine the consecutive wave correlations that were discussed in section 3.3.3. Again we will present mainly numerical work in this section, and will finally verify the various exponent relations presented and derived in section 3.3.3.

Let us start by validating the scaling forms (3.35) and (3.37). Again trying to make collapse plots by trial and error is a tedious work and usually does not yield accurate values for the exponents. Therefore we obtained rather accurate data for the exponents in (3.39)-(3.40) in another way, as will be described below. Using these values we can present collapse plots of the conditional probability distribution functions (3.39)-(3.40). In figure 4.17 the forward conditional probability distribution function $P(s_{k+1} \mid s_k)$
obtained from numerical simulation of the BTW model on a Sierpinski gasket of generation 9 is presented. These data were obtained after initiating $10^7$ avalanches. The data in figure 4.17 are a result of binning the original data in two ways: different curves present different values for $s_k$, binned by using binning intervals delimited by powers of 2, and within each curve each point represents values of $s_{k+1}$ by using binning intervals delimited by powers of $\sqrt{2}$. In figure 4.18 the backward conditional probability distribution function $P(s_k \mid s_{k+1})$ is presented under the same conditions as for figure 4.17. These finite system size data do certainly not contradict the scaling forms (3.39)-(3.40).

Without a priori knowing accurate exponent values, making a plot like in figures 4.17 and 4.18 is not easy. Inspired by our results on waves we will examine the moments of (3.39)-(3.40). Since for e.g. $s_{k+1} \gg s_k$, $P(s_{k+1} \mid s_k) \sim s_{k+1}^{-\beta - r}$, and then the moments of $P(s_{k+1} \mid s_k)$ for $s_{k+1} \gg s_k$ should scale as

$$L^{\sigma^+(q)}$$

with $\sigma^+(q)$ a piecewise linear function as in figure 2.2 or 2.3, from which $\beta + r$ can be determined. In case $s_{k+1} \ll s_k$, $P(s_{k+1} \mid s_k) \sim s_{k+1}^{-\beta}$, and the moments of $P(s_{k+1} \mid s_k)$ for $s_{k+1} \ll s_k$ should scale as

$$L^{\sigma^-(q)}$$

with $\sigma^-(q)$ again a piecewise linear function, from which $\beta$ can be determined. Analogously $P(s_{k} \mid s_{k+1}) \sim s_k^{-\beta}$ for $s_k \ll s_{k+1}$ and $P(s_{k} \mid s_{k+1}) \sim s_k^{-\beta - r}$ for $s_k \gg s_{k+1}$ and the moments of these distributions should scale with $L$ by the moment scaling functions $\tilde{\sigma}^-(q)$ and $\tilde{\sigma}^+(q)$ respectively. From these $\tilde{\beta} + \tilde{r}$ and $\tilde{\beta}$ can be determined.

The functions $\sigma^-(q)$ and $\sigma^+(q)$ obtained from our numerical data are presented in figure 4.19. The indication for finite size scaling behaviour is obvious because already for the rather small system sizes the piecewise linear behaviour is well approximated. The slope indicates a cut-off exponent equal to $D_f$ as could be expected. In figure 4.20 the functions $\tilde{\sigma}^-(q)$ and $\tilde{\sigma}^+(q)$ are presented under analogous conditions. Again the cut-off exponent is $D_f$. From the location of the crossover point in the linear behaviour various exponents can be determined, namely:

$$\beta = 0.35 \pm 0.05$$ (4.56)
Figure 4.17: Collapsed forward conditional probability distribution function $P(s_{k+1} \mid s_k)$ obtained from numerical simulation of the BTW model on a Sierpinski gasket of generation $n = 9$. The data was binned as well over $s_k$ as over $s_{k+1}$ (see text). The collapse of data was obtained using (4.56)-(4.57).

Figure 4.18: Collapsed backward conditional probability distribution function $P(s_k \mid s_{k+1})$, under the same conditions as figure 4.17. The collapse of data was obtained using (4.58)-(4.59).
Figure 4.19: Moment scaling functions $\sigma^-(q)$ (diamonds) and $\sigma^+(q)$ (triangles) for $P(s_{k+1} \mid s_k)$ in the limits $s_{k+1} \gg s_k$ and $s_{k+1} \ll s_k$ respectively.

Figure 4.20: Moment scaling functions $\bar{\sigma}^-(q)$ (diamonds) and $\bar{\sigma}^+(q)$ (triangles) for $\bar{P}(s_k \mid s_{k+1})$ in the limits $s_k \gg s_{k+1}$ and $s_k \ll s_{k+1}$ respectively.
\[ \beta + r = 1.30 \pm 0.03 \quad (4.57) \]
\[ \bar{\beta} = 0.16 \pm 0.05 \quad (4.58) \]
\[ \bar{\beta} + \bar{r} = 1.24 \pm 0.03. \quad (4.59) \]

With these values the various exponent relations derived in section 3.3.3 can be verified. Comparing (4.59) with (4.51) we see that relation (3.41) holds nicely. From (4.56)-(4.59) \( r \) and \( \bar{r} \) can be calculated:

\[ r = 0.95 \pm 0.08 \quad (4.60) \]
\[ \bar{r} = 1.08 \pm 0.08 \quad (4.61) \]

what is consistent with the prediction (3.47). Using (4.50) and (4.51), (3.48) gives the prediction \( \beta = 0.288 \ldots \) This is not too far from the numerically determined result on a rather small lattice (4.56). Also (3.46) seems not to be violated.

So in conclusion there is a rather good correspondence between analytical predictions and numerical results in this study of consecutive wave statistics on the Sierpinski gasket. In addition to similar work on the two dimensional square lattice [50] this allows us to conclude that waves of topplings are now rather well understood at this level of pairs of consecutive waves.

Again the moment analysis method proved its efficiency in extracting clear results from probability distribution functions with rather irregular shapes.

### 4.3.4 Avalanche statistics

We now come to the avalanches of the BTW model on the Sierpinski gasket. As explained in section 2.3.1 finite size scaling behaviour was originally expected in the BTW model (this was one of the main reasons that the model was introduced). Therefore we will examine probability distribution functions for various avalanche properties. In this thesis we restrict ourselves to the avalanche area \( a \), the avalanche size \( s \), and the avalanche duration \( t \). The results gathered from computer simulations for the avalanche area and the avalanche duration distributions are presented in figures 4.21 and 4.22.

As was the case for the wave and last wave probability distribution functions we notice an irregular behaviour superposed on decaying functions that show log-periodic oscillations (however the functions for the avalanche duration turn out to be rather smooth). The peak in the area distribution
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Figure 4.21: Probability distribution function of avalanche area in the BTW model on the Sierpinski gasket.

Figure 4.22: Probability distribution function of avalanche duration in the BTW model on the Sierpinski gasket.
function at the maximum area value is explained by the special character of the sandpile model on the Sierpinski gasket. Sand necessarily has to leave the system at the corner sites. This enforces system-spanning avalanches. The size distribution (not displayed here) shows a similar feature.

Now we present the results of the moment analysis for these probability distribution functions. The moment scaling functions $\sigma_a(q)$, $\sigma_s(q)$ and $\sigma_t(q)$ are shown in figure 4.23. They seem to indicate a finite size scaling for the probability distribution functions. The local slope of these moment scaling functions is shown in figure 4.24.

From these figures we find

$$D_a = 1.57 \pm 0.01 \quad (4.63)$$
$$D_s = 3.04 \pm 0.01 \quad (4.64)$$
$$D_t = 1.18 \pm 0.01 \quad (4.65)$$

and

$$\tau_a = 1.20 \pm 0.03 \quad (4.66)$$
$$\tau_s = 1.26 \pm 0.03 \quad (4.67)$$
$$\tau_t = 1.32 \pm 0.03. \quad (4.68)$$

For these avalanche related exponents there are no exact values with which we can compare. However it is obvious that $D_a = D_f$, the fractal dimension of the Sierpinski gasket, since the avalanche area is limited by the number of sites in the gasket.

Again the results (4.66)-(4.68) obtained by the moment analysis are quite different to results obtained by the collapse method [58, 59]. The experience with the general and last waves, where we could compare with exact results, learned to trust the results from moment analysis rather than those of the collapse method.

Now if finite size scaling holds here there should exist relations like (2.55) between $a$, $s$ and $t$. Furthermore there should exist an invariant $\zeta$ (2.57). Define ad hoc

$$\zeta_x \equiv D_x(\tau_x - 1) \quad \text{with} \quad x \in \{a, s, t\}.$$

From the numerical data (4.63)-(4.65) and (4.66)-(4.68) we find for these quantities
4.3. DYNAMICAL ASPECTS AND SCALING

Figure 4.23: Moment scaling functions $\sigma_{a}(q)$ (squares), $\sigma_{s}(q)$ (diamonds) and $\sigma_{t}(q)$ (triangles) for respectively avalanche area, size and duration in the BTW model on the Sierpinski gasket, computed from the simulation data.

Figure 4.24: Local slope $\Delta \sigma / \Delta q$ of the moment scaling functions $\sigma_{a}(q)$ (squares), $\sigma_{s}(q)$ (diamonds) and $\sigma_{t}(q)$ (triangles) for respectively avalanche area, size and duration in the BTW model on the Sierpinski gasket, computed from the simulation data.
Figure 4.25: Fraction of dissipating avalanches in the BTW model on the Sierpinski gasket for gaskets of generations 5 to 9.

\[
\begin{align*}
\zeta_a &= 0.33 \pm 0.05 \quad (4.69) \\
\zeta_s &= 0.80 \pm 0.05 \quad (4.70) \\
\zeta_t &= 0.38 \pm 0.05. \quad (4.71)
\end{align*}
\]

The difference between these values is striking. They should be much closer to each other in case of finite size scaling. It would be useful if there would be any other information about the value of \( \zeta \). In [35] it is argued that the fraction of dissipative avalanches should scale as

\[
p_{\text{diss}} \sim L^{-\zeta}. \quad (4.72)
\]

In our numerical simulations we also kept track of the dissipation of sand within avalanches. From this we calculated the fraction of avalanches that dissipate sand (these are the avalanches that reach the border) with respect to all avalanches in the simulation. This number, \( p_{\text{diss}} \), is presented in figure 4.25 as a function of the system size \( L \). From these data we find
\[ \zeta = 0.58 \pm 0.05. \] (4.73)

This is an indication that there is a problem with the finite size scaling of the probability distribution functions for the avalanche properties. The fact that all the \( \zeta_x (x = a, s, t) \) are different and also are different from \( \zeta \) indicates that relations like (2.55) cannot exist and hence there can be no simple finite size scaling in the BTW model. (Note however that \( \zeta_a \approx \zeta_t \).)

To further check such a conclusion, let us take a look at the numerical multifractal spectra, presented in figures 4.26, 4.27 and 4.28. To each of these plots an extrapolated \( L \to \infty \) curve is added, as explained in section 4.3.2.

In figure 4.29 the extrapolated spectra for the three avalanche quantities are plotted together. Again we see that there is an accumulation of points in the extrapolated curves. However in the case of finite size scaling this accumulation should occur around \( (D_x, \zeta) \) for each case \( x = a, s, t \). However the spectra accumulate around \( (D_x, \zeta_x) \) and \( \zeta_x \neq \zeta \forall x \in \{a, s, t\} \).

We have to conclude that - although the effect is small - the scaling behaviour of the BTW model on the Sierpinski gasket is more complicated than simple finite size scaling. This conclusion extends De Menech and Stella's work [33, 35] which showed that the same is true on the two dimensional square lattice.

### 4.3.5 Long time correlations of waves

We analysed the correlations within time series of waves by investigating the autocorrelation function \( C_w(t, L) \) defined in (3.34). In figure 4.30 we present this function for Sierpinski gaskets of different generations. We used time series of \( k_{max} = 10^5 \) successive waves. Clearly there are long range correlations between waves, they are even more pronounced than on the two dimensional square lattice [33].

This supports our conjecture of section 3.3.2, namely that strong correlations between waves within avalanches alter the scaling behaviour of the BTW model from finite size scaling to multiscaling.

### 4.4 The Manna model

Here we present an analysis of the probability distribution functions for various avalanche properties in the Manna model on the Sierpinski gasket, namely the avalanche area, size and duration. Conditions for the simulations are similar to those of our simulations of the BTW model. We will denote
Figure 4.26: Finite size multifractal spectrum for the avalanche area distribution in the BTW model on the Sierpinski gasket, computed from the simulation data. The triangles represent an extrapolated curve for $L \to \infty$.

Figure 4.27: Finite size multifractal spectrum for the avalanche size distribution in the BTW model on the Sierpinski gasket, computed from the simulation data. The triangles represent an extrapolated curve for $L \to \infty$. 
Figure 4.28: Finite size multifractal spectrum for the avalanche duration distribution in the BTW model on the Sierpinski gasket, computed from the simulation data. The triangles represent an extrapolated curve for $L \to \infty$.

Figure 4.29: Extrapolated multifractal spectra for avalanche area, size and duration distribution in the BTW model on the Sierpinski gasket.
Figure 4.30: Autocorrelation function for waves in the BTW model on the Sierpinski gasket. The different curves from bottom to top are for gaskets of generations 1 through 9.

quantities in the Manna model with a similar symbol as for the BTW model but with a bar.

The results obtained from computer simulations for these functions are presented in figures 4.31, 4.32 and 4.33.

Like in the BTW model we notice log periodic oscillations and irregularities in these functions, although they seem to be not so pronounced as in the BTW model.

We apply both the moment analysis and the multifractal analysis method to these probability distributions. First let us take a look at the moment scaling functions for the avalanche area, size and duration. These are presented in figure 4.34. From figure 4.35 we see that the local slope of these moment scaling functions becomes constant for $q \geq 2$. This is already a first indication of finite size scaling.

From the moment analysis we find

\[
\begin{align*}
\bar{D}_a &= 1.58 \pm 0.01 \\
\bar{D}_s &= 2.68 \pm 0.01 \\
\bar{D}_t &= 1.63 \pm 0.01
\end{align*}
\]
Figure 4.31: Probability distribution function of avalanche areas in the Manna model on the Sierpinski gasket.

Figure 4.32: Probability distribution function of avalanche sizes in the Manna model on the Sierpinski gasket. The data starting from $s = 16$ for each gasket is binned using binning intervals delimited by powers of $\sqrt{2}$. 
Figure 4.33: Probability distribution function of avalanche durations in the Manna model on the Sierpinski gasket.

Figure 4.34: Moment scaling functions for avalanche area (diamonds), size (triangles) and duration (squares) of the Manna model on the Sierpinski gasket.
Figure 4.35: Local slope of the moment scaling functions of figure 4.34.

and

\[
\bar{\tau}_a = 1.25 \pm 0.03 \quad (4.77)
\]
\[
\bar{\tau}_s = 1.17 \pm 0.03 \quad (4.78)
\]
\[
\bar{\tau}_t = 1.24 \pm 0.03. \quad (4.79)
\]

There are no exact values with which we can compare. However again it is obvious that \( \bar{D}_a = D_f \), the fractal dimension of the Sierpinski gasket, since the avalanche area is limited by the number of sites in the gasket.

If finite size scaling holds here there should exist relations like (2.55) between \( a, s \) and \( t \). Furthermore there should be an invariant \( \bar{\zeta} \) (2.57). Define ad hoc

\[
\bar{\zeta}_x = \bar{D}_x(\bar{\tau}_x - 1) \quad \text{with} \quad x \in \{a, s, t\}.
\]

From the numerical data (4.74)-(4.76) and (4.77)-(4.79) we find for these quantities

\[
\bar{\zeta}_a = 0.40 \pm 0.05 \quad (4.80)
\]
\[
\bar{\zeta}_s = 0.45 \pm 0.05 \quad (4.81)
\]
\[
\bar{\zeta}_t = 0.40 \pm 0.05. \quad (4.82)
\]
Figure 4.36: Fraction of dissipating avalanches in the Manna model on the Sierpinski gasket.

Again we can determine $\bar{\zeta}$ independently through the fraction of dissipative avalanches that should scale as

$$\bar{p}_{diss} \sim L^{-\bar{\zeta}}.$$ 

This fraction obtained from our simulations is presented as a function of the system size $L$ in figure 4.36. From these data we find

$$\bar{\zeta} = 0.44 \pm 0.05.$$ \hspace{1cm} (4.83)

The correspondence between (4.80), (4.81), (4.82) and (4.83) allows us to conclude that the probability distribution functions for avalanches in the Manna model on the Sierpinski gasket obey finite size scaling.

We can check this conclusion by performing a multifractal analysis of these probability distribution functions.

The multifractal spectra were calculated from the numerical obtained probability distribution functions using (2.65)-(2.66). The results are presented in figures 4.37, 4.38 and 4.39, together with an extrapolated $L \to \infty$
Figure 4.37: Finite size multifractal spectrum for the avalanche area distribution in the Manna model on the Sierpinski gasket, computed from the simulation data. The triangles represent an extrapolated curve for $L \to \infty$.

Figure 4.38: Finite size multifractal spectrum for the avalanche size distribution in the Manna model on the Sierpinski gasket, computed from the simulation data. The triangles represent an extrapolated curve for $L \to \infty$. 

Figure 4.39: Finite size multifractal spectrum for the avalanche duration distribution in the Manna model on the Sierpinski gasket, computed from the simulation data. The triangles represent an extrapolated curve for $L \to \infty$.

Figure 4.40: Extrapolated multifractal spectra for avalanche area (squares), size (triangles) and duration (circles) distribution in the Manna model on the Sierpinski gasket. The dashed lines indicate the ideal $L \to \infty$ spectra in case of finite size scaling.
spectrum. This was obtained in the same way as in the BTW case (see section 4.3.2).

Figure 4.40 collects the three extrapolated spectra. Indeed the three spectra pile up around \( \bar{\zeta} \) for increasing \( q \). This affirms our conclusion that the Manna model obeys finite size scaling on the Sierpinski gasket, as it does on the two dimensional square lattice [33, 35].

As another conclusion, we can state that the BTW model and the Manna model do not belong to the same universality class. Indeed the scaling behaviour in the BTW model is more complex than the rather simple finite size scaling in the Manna model.
Chapter 5

The BTW model with dissipation

5.1 Introduction

In addition to the previous study of the BTW and Manna models and of their scaling behaviour on the Sierpinski gasket in particular, we would like to add a study of a nondissipative version of the BTW (or Abelian sandpile) model. We will first present some general analytical results and then again perform concrete calculations on the Sierpinski gasket. Part of this work has been published in [60].

A question which is still poorly understood is what precisely are the necessary ingredients a system (or a model) must have for it to become self-organised critical. For example, one may ask whether or not dissipation of ‘energy’ (or a similar quantity) destroys long range correlations. This problem has been studied extensively in the Olami-Feder-Christensen (OFC) model of earthquakes [4]. Numerical studies originally seemed to show convincingly that even in the presence of a small amount of dissipation the model remains critical [61]. Later, it was shown exactly that at least in mean field, the OFC model is only critical when its energy is conserved [62]. Most recently it was argued on the basis of a study of branching rates, that the same is true on a finite dimensional lattice [63].

The role of conservation of sand in sandpile models was first studied numerically by Ghaffari et al. [64], who found that any amount of dissipation destroys the presence of self-organised criticality in the BTW model. Recently, it was proven that indeed on any hypercubic lattice a non-conservative Abelian sandpile model is not critical [65]. There exists therefore a correlation length $\xi$ in the system which diverges when the dissipation rate goes
to zero. This allows the introduction of an exponent $\nu$ which describes this divergence. Numerically [64] it was found that $\nu \approx 1/2$ in $d = 2$. The same authors argued on the basis of a renormalisation group calculation that $\nu = 1/2$ on any Euclidean lattice. This result was recently proven exactly [66].

In this chapter we study further the Abelian sandpile model with dissipation. We begin by showing that this problem can be related to that of a suitably defined random walker on a lattice with a trap. This result is quite general and extends the earlier mapping between conservative sandpiles and resistor networks (or equivalently the $q \to 0$ Potts model), presented in chapter 3. Indeed, we will show that sending the dissipation to zero is equivalent to taking the long time limit of the random walk problem. It therefore comes as no surprise that the (sandpile) correlation length exponent $\nu$ can be related to the random walk exponent $D_{rw}$ (2.52). Our result implies that $\nu = 1/2$ for any Euclidean lattice. We thus recover in this situation the conclusion of [64, 66] but add a new understanding of it. However our prediction is more general and holds also on, for example, fractal lattices, or for certain types of random sandpiles. As an example, we performed calculations on the Sierpinski gasket for which $D_{rw}$ is known exactly (4.4). Our data are consistent with the prediction $\nu = 1/D_{rw}$, see section 5.3.

## 5.2 Mapping on a random walk problem

In the following we will use a superscript $c$ for the case of the conservative model and $d$ for the case of the dissipative model. As an example, we will denote the toppling matrix of the conservative model (1.1) as $\Delta^c$. We will omit the superscript expression which are valid in both cases. For the toppling matrix of the dissipative sandpile model we follow Tsuchiya and Katori [65]:

$$
\Delta^d_{ij} = \begin{cases} 
  z \gamma \zeta & \text{if } i = j \\
  -\zeta & \text{i and j are neighbours} \\
  0 & \text{otherwise}
\end{cases}
$$

(5.1)

with $\gamma > 1$. In this way, at each toppling $\zeta z(\gamma - 1)$ grains of sand disappear.

In the following we will follow an analogous derivation as in section 2.2.3 using the master equation for the evolution of the probability $P(i,k,t)$ that a random walker is at a site $i$ at time $t$ given that he was in $k$ at time $t = 0$.

With each toppling matrix $\Delta_{ij}$ we can associate a random walk problem. To do this we have to extend the graph $G$ with one extra site, denoted
as $T$ which, as we will see immediately, will get the properties of a trap
for the random walker. Let $G^* = G \cup T$. We then define a continuous
time random walker on $G^*$ using the matrix elements $\Delta_{ij}$ as transition rates. More
concrete, the rate by which the walker jumps from $j$ to $i$ is given by $-\Delta_{ij}$ for
any two sites on $G$. Secondly, the walker jumps from a site $j \in G$ to the trap
$T$ with rate $\phi_j = \sum_i \Delta_{ij}$. Once the walker reaches $T$, it stays there forever.

From this definition we see that in the conservative case this trap site
is nothing but the sink site introduced earlier in the conservative Abelian
sandpile model.

The conditional probability $P(i, k, t)$ evolves according to the master
equation (2.36), but now with

$$D_{ij} = \begin{cases} 
\Delta_{ij} & i \in G, j \in G, \ i \neq j \\
-\phi_j & i = T, j \in G \\
0 & \text{if } j = T
\end{cases} \quad (5.2)$$

For the diagonal elements we take

$$D_{jj} = -\sum_{i \in G, i \neq j} D_{ij} = -\sum_{i \in G, i \neq j} \Delta_{ij} + \phi_j = \delta_{jj}$$

In this way, and because of the conditions we put on the matrix $\Delta$, $D$
has all the necessary properties (see e.g. [67]) of a stochastic matrix.

To solve the master equation (2.36), it is common [67] to introduce the
Laplace transform (2.37) of $P(i, k, t)$. This function obeys the linear equation
(2.38) whose formal solution is (2.39). This solution can be written in terms
of the eigenvalues $\omega_\alpha$ and associated eigenvectors $v_\alpha$ of $D$ as

$$G(i, k, s) = \sum_\alpha \frac{1}{(s + \omega_\alpha)} (v_\alpha)_i (v_\alpha)_k \quad (5.3)$$

Notice that because of the structure (5.1), the spectrum of the matrix
consists of the spectrum of $\Delta$ and one zero eigenvalue, which we will denote
as $\lambda_0$. The eigenvector associated with this zero eigenvalue is completely
concentrated on the trap, $(v_0)_i = \delta_{i,T}$. On the other hand, for the eigenvectors
associated with the other eigenvalues, we have $(v_\alpha)_T = 0, \alpha \neq 0$. Therefore
(5.3) becomes

$$G(i, k, s) = \frac{1}{s} \delta_{i,T} \delta_{k,T} + \tilde{G}(i, k, s)$$
CHAPTER 5. THE BTW MODEL WITH DISSIPATION

where

\[ \tilde{G}(i, k, s) = \sum_{\alpha \neq 0} \frac{1}{(s + \omega_\alpha)} (v_\alpha)_i (v_\alpha)_k \]  \hspace{1cm} (5.4)

We can therefore relate the matrix elements \( G_{ij} \) of the lattice Green function to the elements of the Laplace transform as

\[ G_{ij} = \tilde{G}(i, j, s = 0) \]  \hspace{1cm} (5.5)

We now have all the necessary ingredients to discuss the dissipative sandpiles whose toppling matrix is given in (5.1). Comparing (1.1) with (5.1), we immediately see that

\[ \Delta^d_{ij} = \zeta \Delta^c_{ij} + z \zeta (\gamma - 1) \delta_{ij} \]  \hspace{1cm} (5.6)

This simple relation implies that the eigenvectors of \( \Delta^d \) and \( \Delta^c \) are the same and that their eigenvalues are trivially related by a multiplicative and an additive constant. Therefore we get for the inverse of \( \Delta^d_{ij} \) using (5.5) and the general definition (5.4)

\[ G^d_{ij} = \tilde{G}^d (i, j, s = 0) = \frac{1}{\zeta} \tilde{G}^c (i, j, s = z(\gamma - 1)) \]  \hspace{1cm} (5.7)

This is our main result. It shows the relation between the dissipative sandpile model and the random walker associated with the conservative sandpile model but at \( s = z(\gamma - 1) \). Taking the conservative limit \( \gamma \to 1 \) then corresponds precisely to taking the limit \( t \to \infty \) in the random walk problem. Since that asymptotic limit is determined by the scaling exponent \( D_{rw} \) of the random walk, it can already be expected that the correlation length exponent \( \nu \) is related to \( D_{rw} \).

The precise connection between the two exponents can be obtained as follows. We will calculate \( \langle s \rangle \) as given in (2.47) for the dissipative sandpile defined in (5.1). Summing over \( i \), using (5.7) and (2.37), we get

\[ \sum_{i \in G} G^d_{ik} = \frac{1}{\zeta} \int_0^\infty P^c_0(k, t) e^{-z(\gamma - 1)t} dt \]  \hspace{1cm} (5.8)

where \( P^c_0(k, t) = \sum_{i \in G} P(i, k, t) \) is the probability that the walker that started at \( k \) has not yet been trapped at time \( t \). To calculate \( P^c_0(k, t) \) it is
important to first consider the type of random walk that we have to investigate. Because of the relation (5.7), we have to work with the random walker associated with the matrix \(\Delta^c\) of the conservative sandpile. The resulting random walker is therefore such that only boundary sites, which in the conservative case are the only ones where sand leaves the system, are connected with the trap. Random walks of this type, at least on an Euclidean lattice are easy to study. We will come back to that below.

First, we consider however the limit \(N \to \infty\). In that limit a random walker starting on a typical site will not be trapped for any finite \(t\), since only the boundary sites at infinity are connected with the trap. Therefore, one has \(P_0^c(k, t) = 1\). Then the integral in (5.8) can immediately be performed and since the result does not depend on \(k\) we get

\[
\langle s \rangle = \frac{1}{\zeta z(\gamma - 1)} \quad (5.9)
\]

This relation was first derived on the \(d = 2\) square lattice in [65] but we now see that it is quite general. In fact, it shows that the dissipative sandpile is never critical. Indeed \(\langle s \rangle\) should diverge as \(N \to \infty\) in order for the model to be critical.

We now turn back to the case that the number of sites on the graph is finite, in which case \(P_0^c(k, t)\) will be a decreasing function of time. Explicit results for this quantity can be obtained on Euclidean lattices with elementary Fourier techniques. In one dimension one obtains for example

\[
P_0^c(k, t) = \frac{2}{L + 1} \sum_{j=1}^{L} \sum_{n=1}^{L} e^{-\omega_n t} \sin \left( \frac{n\pi k}{L + 1} \right) \sin \left( \frac{n\pi j}{L + 1} \right)
\]

where \(\omega_n = 2(1 - \cos(n\pi/(L + 1)))\). Similar results can easily be obtained in higher dimensions. In the scaling limit, \(L \to \infty, t \to \infty\), it is easy to see that this probability is of the form

\[
P_0^c(k, t) \approx \frac{1}{L} F \left( \frac{k}{L}, \frac{t}{L^2} \right) \quad (5.10)
\]

On any Euclidean lattice, the fractal dimension \(D_{ru}\) of the type of random walker that we consider here equals 2. On the basis of exact results such as (5.10), and on general physical intuition it can be expected that for the random walkers which are connected with the trap only through some boundary sites, \(P_0^c(k, t)\) has in general the following scaling behaviour
\[ P_0^c (k, t) \approx \frac{1}{L^{D_f}} H \left( \frac{k}{L}, \frac{t}{L^{D_{rw}}} \right) \]  \hspace{1cm} (5.11)

with \( H \) some scaling function. Here \( D_f \) is the (fractal) dimension of the graph.

After inserting (5.11) in (5.8), making a suitable change of variables, and also performing the resulting sum over \( k \) we finally get the following scaling form for \( \langle s \rangle \)

\[ \langle s \rangle \sim L^{D_{rw}} R \left( L(\gamma - 1)^{1/D_{rw}} \right) \]  \hspace{1cm} (5.12)

where \( R \) is a scaling function.

(5.12) coincides with (3.31) in the conservative case (\( \gamma = 1 \)). For \( \gamma > 1 \), we conclude from (5.12) that the exponent \( \nu \) that describes the crossover between dissipative and conservative sandpiles equals

\[ \nu = \frac{1}{D_{rw}} \]  \hspace{1cm} (5.13)

On an Euclidean lattice, we recover in this way, as mentioned earlier, a result first determined with an approximate renormalisation technique [64] and recently obtained exactly [66].

### 5.3 The Sierpinski gasket revisited

Our result (5.12) is however much more general. The relation (5.13) should also hold on e.g. fractal lattices. As an example, we checked this scaling on finite Sierpinski gaskets. In particular, we considered Sierpinski gaskets of \( n \) generations with \( n \leq 7 \). For each of these we calculated the matrix \( G^d_{ij} \) by calculating the inverse of \( \Delta^d_{ij} \) using a computer. This we did for several values of \( \gamma \). From \( G^d \) we then calculated \( \langle s \rangle \) using (2.47). Finally, we plotted \( \langle s \rangle L^{-D_{rw}} \) versus \( L(\gamma - 1)^{1/D_{rw}} \). These numerically exact results are shown in figure 5.1. For the Sierpinski gasket \( D_{rw} \) is given by (4.4). As can be seen the agreement with the scaling prediction is excellent. Together with the known result for the square lattice case, these data give very strong support for the conjecture (5.13).
5.4 The exponent $y_H$ on the Sierpinski gasket

In contradiction to our conjecture (5.13) Katori made the conjecture [68] that $\nu$ would equal $1/y_H$, where $y_H$ is the critical exponent of the Potts model related to the interaction with an external magnetic field. This is certainly true in two dimensions since there $y_H = 2$. The hypothesis can now be checked on the Sierpinski gasket since following our argumentation we believe (5.13) should hold and thus on the Sierpinski gasket $y_H$ should equal $D_{rw}$.

Inspired by our work of section 4.2.1 we can determine $y_H$ exactly on the Sierpinski gasket via a renormalisation group calculation. Therefore we have to extend the Potts Hamiltonian (4.15) with a term that describes the interaction of the spins with an external magnetic field. Let us consider an external magnetic field that prefers the spins to take on one specific value, let us take $q = 1$. The Potts Hamiltonian then becomes

$$-\mathcal{H}(\vec{K}, q) = \sqrt{q} K \sum_{(ij) \in E(G)} \delta_{\sigma_i, \sigma_j} + q L \sum_k \delta_{\Delta k} + H q^\alpha \sum_{l \in V(G)} \delta_{\sigma_l, 1}$$

(5.14)

with $\vec{K} = (K, L, H)$, and at the moment there is no indication to what $\alpha$ should be. We will now perform a renormalisation procedure to calculate the
partition function for this model on the Sierpinski gasket, in an analogous way as was done in section 4.2.1 for the thermal interactions only.

First we notice that the spin-field interaction term will generate several new kinds of interaction terms during the renormalisation, namely:

\[
M q^\beta \sum_{(ij) \in \mathcal{E}(G)} \delta_{\sigma_i,1} \delta_{\sigma_j,1} ,
\]

\[
R q^\gamma \sum_{(i,j,k) \in \Delta \atop k \notin \{i,j\}} \delta_{\sigma_i,1} \delta_{\sigma_j,1} \delta_{\sigma_k,1} ,
\]

\[
S q^\delta \sum_{(i,j,k) \in \Delta} \delta_{\sigma_i,1} \delta_{\sigma_j,1} \delta_{\sigma_k,1} .
\]

where $\Delta$ stands for the upright triangles in the lattice. We will use the notations

\[
\delta_{\sigma_i,\sigma_j,1} \equiv \delta_{\sigma_i,1} \delta_{\sigma_j,1} ,
\]

\[
\delta_{\Delta,1} \equiv \delta_{\sigma_i,1} \delta_{\sigma_j,1} \delta_{\sigma_k,1} ,
\]

and again stress that three-spin interactions only appear within one (upright) triangle $\Delta$. At this moment we don’t know what $\beta$, $\gamma$ and $\delta$ should be. The total Hamiltonian to start the renormalisation with is

\[
-\mathcal{H}(\vec{K}, q) = \sqrt{q} K \sum_{(ij) \in \mathcal{E}(G)} \delta_{\sigma_i,\sigma_j} + q L \sum_k \delta_{\Delta_k} + H q^\alpha \sum_{r \in \mathcal{V}(G)} \delta_{\sigma_{r,1}} + M q^\beta \sum_{(mn) \in \mathcal{E}(G)} \delta_{\sigma_{m,1}} + R q^\gamma \sum_{(r,s,t) \in \Delta \atop t \notin \{r,s\}} \delta_{\sigma_{r,1}} \delta_{\sigma_{s,1}} + S q^\delta \sum_u \delta_{\Delta_{u,1}}
\]

with $\vec{K} = (K, L, H, M, R, S)$. We proceed in the same way as in section 4.2.1, and the expression analogous to (4.18) is now

\[
\prod_{(ij)} \left(1 + \sqrt{q} K \delta_{\sigma_i,\sigma_j}\right) \prod_{k=1}^3 \left(1 + q L \delta_{\Delta_k}\right) \prod_{l=1}^3 \left(1 + \frac{1}{2} q^\alpha H \delta_{\sigma_{l,1}}\right) \prod_{m=4}^6 \left(1 + q^\alpha H \delta_{\sigma_{m,1}}\right) \prod_{(nr)} \left(1 + q^\beta M \delta_{\sigma_{n,1}}\right)
\]
within an arbitrary elementary triangle (products over \(\langle ab \rangle\) run over all bonds \(\sigma_a - \sigma_b\) in an elementary triangle). Note that we split up the spin-field interaction for the corner spins and the inner spins. Each corner spin \(i\) of an elementary triangle appears also in one other elementary triangle (except in the lowest generation gasket). Therefore we split up its contribution \(q^a H \delta_{\sigma_i,1}\) over these two elementary triangles, giving each corner spin \(i\) a weight factor \(\frac{1}{2} q^a H \delta_{\sigma_i,1}\) within each elementary triangle.

We have to take the trace of (5.18) over the spins \(\sigma_4, \sigma_5\) and \(\sigma_6\), in the limit \(q \to 0\). We start this calculation by expanding (5.18), which yields a very large number of terms of all possible combinations of the interactions present in (5.17).

The following consideration will substantially simplify the calculations. We can be sure that

\[
\{K^* = L^* = H^* = M^* = R^* = S^* = 0\}
\]

will be a fixed point of the renormalisation equations. Indeed when we did not take into account the spin-field interactions we found a critical point \(\{K^*, L^*\}\) without generating spin-field interactions. Hence this will still be a critical point when in the renormalisation equations all magnetic field parameters are put equal to zero, because then we will find back the equations of the model without the spin-field interactions.

This does say nothing about eventual other fixed points where one or more of the magnetic field parameters differ from zero. Here we will only study the critical behaviour of the Potts model with an external field around the critical point (5.19).

This simplifies the equations enormously because now we only have to find the linear parts of the renormalisation equations. Indeed a nonlinear term will give a zero contribution in the renormalisation matrix \(\partial K_i^*/\partial K_j\). This means that in the expansion of (5.18) we only have to look at terms with a weight factor linear in \(H, M, R\) or \(S\) and disregard those with a higher contribution. The trace over the terms in which these magnetic field parameters do not appear has already been calculated in section 4.2.1.

Since we only need the exponent \(y_H\) it is not necessary to really calculate the partition function in the limit \(q \to 0\). What we need are the renormalisation equations in the limit \(q \to 0\), linearised in the field interaction
parameters, and the associated renormalisation matrix. At this moment we can already see that this matrix should have the following structure:

\[
\mathcal{R} = \begin{pmatrix}
\frac{3}{5} & \frac{29}{250} & 1 & x & x & x \\
0 & \frac{27}{625} & x & x & x & x \\
0 & 0 & x & x & x & x \\
0 & 0 & x & x & x & x \\
0 & 0 & x & x & x & x
\end{pmatrix}
\]

where for the rows and columns we always keep the order \((K, L, H, M, R, S)\). The thermal spin-spin interactions \((K \text{ and } L)\) do not generate the spin-field interactions and therefore the matrix \(\mathcal{R}\) has a block structure. This is very convenient because now we only need to calculate the elements of the 4x4 block associated with the spin-field interactions.

We start the calculation by performing the trace over (5.18). This part of the calculation is presented in appendix A.4. The result for the thermically controlled part is the same as (4.23). The additional terms that contain the field interaction parameters are (with a factor \(50K^3q^3\) separated as in (4.25))

\[
50K^3q^3 \left( \left( \frac{75}{50} q^\alpha H + \frac{1}{K} q^{\beta - \frac{1}{2}} M + \frac{1}{K} q^{\gamma - \frac{1}{2}} R + \frac{4}{50} \frac{1}{K^2} q^{\delta - 1} S \right) \sum_{i=1}^{3} \delta_{\sigma_i,1} \\
+ \left( \frac{108}{50} K q^{\alpha + \frac{1}{2}} H + \frac{108}{50} q^{\beta} M + \frac{90}{50} q^{\gamma} R + \frac{12}{50} \frac{1}{K} q^{\delta - \frac{1}{2}} S \right) \sum_{j,k=1 \atop j \neq k}^{3} \delta_{\sigma_j,\sigma_k,1} \\
+ \left( \frac{27}{50} K q^{\alpha - \frac{1}{2}} H + \frac{12}{50} q^{\beta} M + \frac{30}{50} q^{\gamma} R + \frac{1}{50} \frac{1}{K} q^{\delta - \frac{1}{2}} S \right) \sum_{i,m,n=1 \atop i \neq m, n \notin \{1, m\}}^{3} \delta_{\sigma_i,\sigma_m,\sigma_n} \\
+ \left( \frac{243}{50} K^2 q^{\alpha + 1} H + \frac{270}{50} K q^{\beta + \frac{1}{2}} M + \frac{270}{50} K q^{\gamma + \frac{1}{2}} R + \frac{36}{50} q^{\delta} S \right) \delta_{\Delta,1} \right).
\]
The prefactor $50K^3q^{3/2}$ represents the contribution to the partition function of the eliminated spins, as explained on page 83.

(5.21) has to be compared to the analogous expression of (5.18) for the triangle between $\sigma_1$, $\sigma_2$ and $\sigma_3$ in the lattice of generation $n-1$. This expression is

$$
1 + \sqrt{q}K' \sum_{i,j=1 \atop i \neq j}^{3} \delta_{\sigma_i,\sigma_j} \left(1 + qL'\delta_{\Delta} \right)
$$

$$
\left(1 + \frac{1}{2}q^{\alpha}H' \sum_{k=1}^{3} \delta_{\sigma_{k,1}} \right) \left(1 + q^{\beta}M' \sum_{l,m=1 \atop l \neq m}^{3} \delta_{\sigma_{l,\sigma_{m,1}}} \right)
$$

$$
\left(1 + q^{\gamma}R' \sum_{n,r,s=1 \atop n \neq r, s \neq \{n,r\}}^{3} \delta_{\sigma_{n,\sigma_{r,\sigma_{s,1}}} \left(1 + q^{\delta}S'\delta_{\Delta,1} \right). \right.
$$

(5.22)

Note the factor $1/2$ in the factor with the single spin-field interaction. As already explained above this factor is there because the contribution of the spin-field interaction of a corner spin has to be divided over the two triangles (elementary triangles in the gasket of generation $n$ or ordinary triangles in the gasket of generation $n-1$) that it is in.

(5.22) becomes after expansion, retaining only linear terms in the field interaction parameters, and up to the lowest order contribution in $q$ in each kind of interaction term (provided none of the exponents $\alpha$, $\beta$, $\gamma$ and $\delta$ are zero):

$$
1 + \sqrt{q}K' \sum_{i,j=1 \atop i \neq j}^{3} \delta_{\sigma_i,\sigma_j} + q \left(3K'^2 + L' \right)\delta_{\Delta}
$$

$$
+ \frac{1}{2}q^{\alpha}H' \sum_{k=1}^{3} \delta_{\sigma_{k,1}} + \left(q^{\beta}M' + q^{\alpha + \frac{1}{2}}K'H' \right) \sum_{l,m=1 \atop l \neq m}^{3} \delta_{\sigma_{l,\sigma_{m,1}}} \right)
$$

$$
+ \left(q^{\gamma}R' + \frac{1}{2}q^{\alpha + \frac{1}{2}}K'H' \right) \sum_{n,r,s=1 \atop n \neq r, s \neq \{n,r\}}^{3} \delta_{\sigma_{n,\sigma_{r,\sigma_{s,1}}} \left(1 + q^{\delta}S' \right)
$$

$$
+ \frac{3}{2}q^{\alpha + 1} \left(3K'^2 + L' \right)(H' + 6q^{\beta + \frac{1}{2}}K'M' + 6q^{\gamma + \frac{1}{2}}K'R') \delta_{\Delta,1}. \right)
$$

A comparison of the first line of this expression with (4.23) yields the known renormalisation equations (4.28) for $K$ and $L$, while a comparison of
the prefactors of the spin-field interactions yields after some calculations the following renormalisation equations:

\[
\begin{align*}
H' &= 3H + 2\frac{1}{K}q^{q^{-\alpha - \frac{1}{2}}}M + 2\frac{1}{K}q^{q^{-\alpha - \frac{1}{2}}}R + 8\frac{1}{50}K q^{q^{-\alpha - 1}}S \\
M' &= 18\frac{1}{50}K q^{q^{-\alpha + \frac{1}{2}}}H + 48\frac{1}{50}q^{q^{-\beta}}M + 30\frac{1}{50}q^{q^{-\beta}}R + 36\frac{1}{250}K q^{q^{-\beta - \frac{1}{2}}}S \\
R' &= -18\frac{1}{50}K q^{q^{-\alpha - \frac{1}{2}}}H - 18\frac{1}{50}q^{q^{-\beta}}M - 7\frac{1}{250}K q^{q^{-\gamma - \frac{1}{2}}}S \\
S' &= 27\frac{1}{625}S \\
\end{align*}
\]

Finally the part $R_2$ of the renormalisation matrix associated with the spin-field interactions becomes

\[
R_2 = \begin{pmatrix}
3 & 2\frac{1}{K}q^{q^{-\alpha - \frac{1}{2}}} & 2\frac{1}{K}q^{q^{-\alpha - \frac{1}{2}}} & 8\frac{1}{50}K q^{q^{-\alpha - 1}} \\
18\frac{1}{50}K q^{q^{-\alpha + \frac{1}{2}}} & 48\frac{1}{50} & 30\frac{1}{50}q^{q^{-\beta}} & 36\frac{1}{250}K q^{q^{-\beta - \frac{1}{2}}} \\
-18\frac{1}{50}K q^{q^{-\alpha - \frac{1}{2}}} & -18\frac{1}{50}q^{q^{-\beta}} & 0 & -7\frac{1}{250}K q^{q^{-\gamma - \frac{1}{2}}} \\
0 & 0 & 0 & 27\frac{1}{625} \\
\end{pmatrix}
\]

The presence of $q$ and $K$ in the equations (5.23) may seem odd since these are the renormalisation equations in the limit $q \to 0$, and the fixed point value of $K$ is $K^\ast = 0$, which leads to divergences in (5.23). However the relevant properties of the matrix $R_2$ are independent of $q$ and $K$, such as the determinant and eigenvalues. Nevertheless, to make the equations (5.23) independent of $q$, a consistent choice for the exponents should be

\[
\beta = \gamma = \alpha + \frac{1}{2}, \quad \delta = \alpha + 1.
\]

But we are only interested in the critical exponents of the Potts model with external field, on the Sierpinski gasket, in the limit $q \to 0$. The eigenvalues of $R_2$ are

\[
\lambda_3 = 3, \quad \lambda_4 = \frac{3}{5}, \quad \lambda_5 = \frac{9}{25} = \left(\frac{3}{5}\right)^2, \quad \lambda_6 = \frac{27}{625} = \frac{1}{5}\left(\frac{3}{5}\right)^3
\]
and hence the critical exponents are, written in terms of the known exponents related to the Sierpinski gasket,

\[
y_3 = \frac{\log 3}{\log 2} = D_f \\
y_4 = \frac{\log 3}{\log 2} - \frac{\log 5}{\log 2} = D_f - D_{rw} \\
y_5 = 2 \left( \frac{\log 3}{\log 2} - \frac{\log 5}{\log 2} \right) = 2(D_f - D_{rw}) \\
y_6 = 3 \frac{\log 3}{\log 2} - 4 \frac{\log 5}{\log 2} = 3D_f - 4D_{rw}.
\]

\(\lambda_3\) is the only relevant eigenvalue, and hence \(y_3\) the only relevant exponent (the other ones are negative). Hence we conclude that

\[
y_H = D_f
\]

for the \(q \to 0\) Potts model on the Sierpinski gasket, for the critical point \(K^+ = 0\).

As a conclusion, according to our data fit in figure 5.1 of the scaling relation (5.12) (which was based mainly on the scaling assumption (5.11)), we have a strong indication that \(\nu = 1/D_{rw}\) on the Sierpinski gasket. Hence we have shown that the conjecture \(\nu = 1/y_H\) [68] does not hold on the Sierpinski gasket, since there \(y_H = D_f\). Katori's conjecture is therefore not correct and should be replaced with (5.13).

This concludes our research of sandpile models on fractal lattices.
Appendix A

Renormalisation of the Potts model on the Sierpinski gasket

A.1 Lowest order 3 spin interaction generated by the 2 spin interactions

What we need to calculate is lowest order term in $q$ with a 3 spin interaction that is generated by mere 2 spin interaction terms in one renormalisation step within an elementary triangle.

It is clear that to build a 3 spin interaction in the next generation gasket from 2 spin interactions in the current generation, at least four different 2 spin interaction terms are needed. Each 2 spin interaction has a 'weight' $p$, as can be seen from (4.12). Since, as $q \to 0$, $p \sim \sqrt{q}$, the generated 3 spin interaction will at least be of order $q^2$. However the term with four 2 spin interactions will always leave one spin free, which will yield an extra factor $q$ when performing the trace. Hence the 3 spin interaction term of lowest order in $q$ generated by four 2 spin interaction terms will be of order $q^3$.

Five 2 spin interaction terms will have a common weight of order $q^{\frac{3}{2}}$ and can leave no spin free. Thus the 3 spin interaction terms generated by five 2 spin interaction terms will be of a lower order in $q$ than those generated by four 2 spin interaction terms.

3 spin interaction terms generated by more 2 spin interaction terms will also always be of higher order in $q$. We conclude that the lowest order 3 spin interaction is generated by five 2 spin interactions and it will be of order $q^{\frac{3}{2}}$.

But, we have to write the renormalised Hamiltonian in a form similar to the original Hamiltonian. As can be seen from (4.25) we therefore have to isolate a factor proportional to $q^{\frac{3}{2}}$ from the renormalised Hamiltonian,
because it is necessary to calculate the contribution of the eliminated spins in the renormalised Hamiltonian.

Hence we see that the lowest order 3 spin interaction terms generated by 2 spin interaction terms are of order $q$, i.e. of order 1.

**A.2 Calculation of the trace within an ET**

Here we will illustrate the calculation of the trace over (4.18) within an elementary triangle, over the 'inner' spins ($\sigma_4, \sigma_5$ and $\sigma_6$).

For further reference, let us enumerate the bonds within an elementary triangle as in figure A.1. Additionally we will use the numbering of sites presented in figure 4.3 and of triangles presented in figure 4.5.

Denote the first product in (4.18) by $A$, and the second by $B$. We have to expand $A$ and $B$ and multiply each term in $A$ with each term in $B$.

Thereby, each 2 spin interaction in $A$ contributes a factor $\sqrt{q}K$, and each 3 spin interaction in $B$ contributes a factor $qL$. And each free spin or each cluster of spins interconnected by interaction terms contributes an extra factor $q$ when performing the trace. However the corner spins will be assumed random but fixed, i.e. they can not be considered as free spins, and each cluster of spins connected to at least one of them will not contribute a factor in $q$ when performing the trace. This is because we perform the trace in the special way of (4.9), namely generation after generation.

Since we work in the limit $q \to 0$, it suffices to retain only the lowest order term in $q$ within each interaction term in the final result. This simplifies the
A.2. CALCULATION OF THE TRACE WITHIN AN ET

calcultations.

It is clear to see that any combination of interaction terms after the expansion of (4.18) can only yield three different types of interaction terms after one renormalisation step, namely: a constant contribution, a 2 spin interaction term, or a 3 spin ('upright triangle') interaction term.

The expansion of $A$ yields $2^9 = 512$ different terms.

The expansion of $B$ yields 4 different kind of terms what concerns the order in $q$, (following the notation convention of (4.14) and figure 4.5)

$$B = 1 + qL \sum_{i=1}^{3} \delta_{\Delta_i} + q^2 L^2 \sum_{i,j=1 \atop i \neq j}^{3} \delta_{\Delta_i} \delta_{\Delta_j} + q^3 L^3 \delta_{\Delta_1} \delta_{\Delta_2} \delta_{\Delta_3}. \quad (A.1)$$

We will successively take each kind of term in (A.1) and multiply all terms of this kind with $A$.

1. Constant term (in $B$) - no triangular interactions

We have 512 different terms consisting of any combination of 2 spin (nearest neighbour) interaction terms. We can graphically represent a 2 spin interaction by a bond. Hence we can represent every term by a graph, and look at all different graphs within an elementary triangle, with number of bonds ranging from 0 to 9. We analyse now separately the three possible interactions that can emerge between the lower generation spins after performing the trace.

Taking into account the weight factors that each bond contributes and the extra weight factors coming from free spins or clusters, it does not take much time to see which graphs with what number of bonds will yield the lowest order term within each kind of interaction. Then it rests to write down all possible graphs with this number of bonds and count how many of these graphs do yield this lowest order interaction term. (It is important to see that not all graphs with this number of bonds will yield a good result.) In table A.1 the results of such a calculation are summarised. For each interaction term generated by taking the trace, we mention the number of bonds necessary in the graph to obtain the lowest order contribution, together with the weight in $q$ that these graphs have. In the third row we mention the number of distinct graphs that produce the lowest order contribution. Additional we give some 'good' and some 'bad' examples, by bad we mean examples of graphs with the right number of bonds but which do not generate a lowest order term.
<table>
<thead>
<tr>
<th>Generated interaction:</th>
<th>constant</th>
<th>2 spin</th>
<th>3 spin</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of bonds</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>weight</td>
<td>$q^3 K^3$</td>
<td>$q^2 K^4$</td>
<td>$q^5 K^5$</td>
</tr>
<tr>
<td>number of graphs</td>
<td>50</td>
<td>30*</td>
<td>54</td>
</tr>
<tr>
<td>good examples</td>
<td>$b_1 b_2 b_5$</td>
<td>$b_1 b_4 b_5 b_6$</td>
<td>$b_1 b_4 b_5 b_6 b_7$</td>
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<td></td>
<td>$b_1 b_2 b_9$</td>
<td>$b_1 b_2 b_6 b_8$</td>
<td>$b_1 b_2 b_4 b_6 b_9$</td>
</tr>
<tr>
<td></td>
<td>$b_1 b_7 b_8$</td>
<td>$b_2 b_4 b_5 b_6$</td>
<td>$b_1 b_4 b_7 b_8 b_9$</td>
</tr>
<tr>
<td>bad examples</td>
<td>$b_1 b_3 b_4$</td>
<td>$b_1 b_3 b_5 b_6$</td>
<td>$b_1 b_2 b_3 b_5 b_6$</td>
</tr>
<tr>
<td></td>
<td>$b_2 b_7 b_8$</td>
<td>$b_1 b_5 b_8 b_9$</td>
<td>$b_1 b_2 b_4 b_6 b_7$</td>
</tr>
<tr>
<td></td>
<td>$b_3 b_5 b_6$</td>
<td>$b_1 b_4 b_5 b_8$</td>
<td>$b_1 b_4 b_5 b_8 b_9$</td>
</tr>
</tbody>
</table>

Table A.1: Summary of calculation of the trace over $A$ multiplied by the constant term in $B$ in the limit $q \to 0$. For each kind of interaction term we mention the number of bonds necessary to yield the lowest order contribution, the weight of these graphs and the number of them, together with some 'good' and 'bad' examples of graphs with this number of bonds (see text). $b_i$ stands for bond $i$ following the labelling of figure A.1.

* That is 30 graphs for each of the three different 2 spin interactions.
2. **1 triangular interaction**

We now have $3 \times 512$ terms, consisting of a triangular interaction (for which there are three possibilities) with additionally any combination of 2 spin interaction terms. Again by weight-considerations it is easy to see that only graphs with one additional 2 spin interaction yield the lowest order constant contribution after performing the trace, while only graphs with two additional 2 spin interactions yield the lowest order 2 spin interaction term and only graphs with three additional 2 spin interactions yield the lowest order 3 spin interaction term. Again not all graphs with the appropriate number of bonds will yield the correct lowest order interaction term. The exact number can again be determined by writing out all\(^1\) possible combinations and counting the good ones. The results are summarised in table A.2. For each kind of interaction terms the lowest order corresponds to the lowest order in the previous case (Constant term - no triangular interactions).

3. **2 triangular interactions**

Again we have $3 \times 512$ terms, consisting of two triangular interactions (for which there are three possibilities) with additionally any combination of 2 spin interaction terms. It is again easy to see that all constant contributions generated here will be of a higher order in $q$ than the ones obtained in the previous cases (constant and single triangular interaction). Only the graphs with no additional bonds yield a lowest order 2 spin interaction, and those with one additional bond yield a lowest order 3 spin interaction. The results are in table A.3.

4. **3 triangular interactions**

These will always be at least of order $q^3$ and hence can not contribute to the lowest order terms, since the lowest order term for the constant contribution is of order $q^{\frac{3}{2}}$, the one for the 2 spin interaction is of order $q^2$ and the one for the 3 spin interaction of order $q^\frac{5}{2}$.

In conclusion, the trace over (4.18) within an elementary triangle over the spins $\sigma_4$, $\sigma_5$ and $\sigma_6$ is

---

\(^1\)Of course it is not necessary to really write out all combinations. When one works in a systematic way one only has to write down the combinations with good chance for success.
<table>
<thead>
<tr>
<th>Generated interaction:</th>
<th>constant</th>
<th>2 spin</th>
<th>3 spin</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of additional bonds</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>weight</td>
<td>$q^{3}KL$</td>
<td>$q^{2}K^{2}L$</td>
<td>$q^{5}K^{3}L$</td>
</tr>
<tr>
<td>number of graphs</td>
<td>12</td>
<td>13*</td>
<td>36</td>
</tr>
<tr>
<td>good examples</td>
<td>$\Delta_{1} + b_{5}$</td>
<td>$\Delta_{1} + b_{4}b_{6}$</td>
<td>$\Delta_{1} + b_{4}b_{5}b_{9}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_{1} + b_{7}$</td>
<td>$\Delta_{1} + b_{6}b_{8}$</td>
<td>$\Delta_{1} + b_{5}b_{8}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_{1} + b_{8}$</td>
<td>$\Delta_{1} + b_{7}b_{8}$</td>
<td>$\Delta_{1} + b_{6}b_{7}b_{8}$</td>
</tr>
<tr>
<td>bad examples</td>
<td>$\Delta_{1} + b_{1}$</td>
<td>$\Delta_{1} + b_{1}b_{2}$</td>
<td>$\Delta_{1} + b_{1}b_{3}b_{5}$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_{1} + b_{4}$</td>
<td>$\Delta_{1} + b_{1}b_{5}$</td>
<td>$\Delta_{1} + b_{1}b_{4}b_{8}$</td>
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<td>$\Delta_{1} + b_{7}$</td>
<td>$\Delta_{1} + b_{1}b_{9}$</td>
<td>$\Delta_{1} + b_{1}b_{8}b_{9}$</td>
</tr>
</tbody>
</table>

Table A.2: Summary of calculation of the trace over $A$ multiplied by the single 3 spin interaction terms in $B$ in the limit $q \to 0$. For each kind of generated interaction term we mention the number of bonds necessary to yield the lowest order contribution, the weight of these graphs and the number of them, together with some 'good' and 'bad' examples of graphs with this number of bonds. $\Delta_{i}$ stands for triangle $i$ following the labelling of figure 4.5.

* That is 13 graphs for each of the three different 2 spin interactions.
<table>
<thead>
<tr>
<th>Generated interaction:</th>
<th>constant</th>
<th>2 spin</th>
<th>3 spin</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of additional bonds</td>
<td>-</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>weight</td>
<td>(O(q^2))</td>
<td>(q^2 L^2)</td>
<td>(q^{\frac{5}{2}}KL^2)</td>
</tr>
<tr>
<td>number of graphs</td>
<td>-</td>
<td>1*</td>
<td>6</td>
</tr>
<tr>
<td>good examples</td>
<td>-</td>
<td>(\Delta_1 \Delta_2)</td>
<td>(\Delta_1 \Delta_2 + b_7)</td>
</tr>
<tr>
<td>bad examples</td>
<td>-</td>
<td>-</td>
<td>(\Delta_1 \Delta_2 + b_3)</td>
</tr>
</tbody>
</table>

Table A.3: Summary of calculation of the trace over \(\mathcal{A}\) multiplied by the double 3 spin interaction terms in \(\mathcal{B}\) in the limit \(q \to 0\). For each kind of generated interaction term we mention the number of bonds necessary to yield the lowest order contribution, the weight of these graphs and the number of them, together with some 'good' and 'bad' examples of graphs with this number of bonds.

* That is 1 graph for each of the three different 2 spin interactions.
\[
a(\vec{K})q^2 + b(\vec{K})q^2 \sum_{i,j=1 \atop i \neq j}^3 \delta_{\sigma_i, \sigma_j} + c(\vec{K})q^{5/2} \delta_{\Delta}
\]

(A.2)

where

\[
\begin{align*}
   a(\vec{K}) &= 50K^3 + 12KL \\
   b(\vec{K}) &= 30K^4 + 13K^2L + L^2 \\
   c(\vec{K}) &= 54K^5 + 36K^3L + 6KL^2.
\end{align*}
\]

(A.3)

A.3 Calculation of \( \mathcal{Z}^{(0)} \)

We calculate explicitly the partition function \( \mathcal{Z}^{(0)} \) in the limit \( q \to 0 \) of the Potts model on a Sierpinski gasket of generation \( n = 0 \) with an additional sink site. By definition

\[
\mathcal{Z}^{(0)} = \mathcal{Tr}_{\sigma_i, i \in \mathcal{V}(G)} \ e^{\mathcal{H}}
\]

(A.4)

with \( \mathcal{H} \) here the Potts Hamiltonian on such a gasket. In figure A.2 we present the gasket. During the renormalisation procedure, it is reached by performing \( n \) renormalisation steps upon a gasket of generation \( n \). After each step the Hamiltonian parameters change according to (4.28). However the interactions between the corner spins of the gasket and the sink site have not been renormalised. Hence a 2 spin interaction between a corner site and the sink site has a weight \( \sqrt{qK} \), while an interaction between two corner spins has a weight

\[
K^{(n)} = \left( \frac{3}{5} \right)^n K.
\]

(We put \( L = 0 \) already from the start.) (A.4) becomes

\[
\mathcal{Z}^{(0)} = \mathcal{Tr} \prod_{i=1 \atop j=i+1}^3 \left( 1 + \delta_{\sigma_i, \sigma_j} \sqrt{qK^{(n)}} \right) \prod_{k=1}^3 \left( 1 + \delta_{\sigma_k} \sqrt{qK} \right)^2
\]

(A.5)

where \( s \) indicates the sink site.

After expanding (A.5) we get a series of terms with all possible combinations of 2 spin interactions. A cluster of four interconnected spins can be formed by an interaction term with three 2 spin interactions, which has a weight factor proportional to \( q^{3/2} \). The cluster itself gets an additional factor \( q \) when performing the trace, yielding a factor proportional to \( q^{5/2} \) in total. It is
easy to verify that no other kind of interaction term can yield a contribution of this, or a lower, order in $q$.

Then it is not much work to write down all possible combinations of three different bonds out of the set of nine bonds of the zero'th generation gasket (remember that each corner site is connected to the sink site by two different bonds), and mark all the combinations that connect all the four sites. These turn out to be 50 different combinations. Giving each renormalised bond a weight $\sqrt{q}K^{(n)}$ and the bonds with the sink site a weight $\sqrt{q}K$, and taking into account the extra factor $q$ that comes from the trace, we find

$$Z^{(0)}(\tilde{K}^{(n)}, q) = \left(8K^3 + 24K^2K^{(n)} + 18K\left(K^{(n)}\right)^2\right)q^{\frac{n}{2}} \quad (A.6)$$

$$= \left(8 + 24\left(\frac{3}{5}\right)^n + 18\left(\frac{3}{5}\right)^{2n}\right)q^{\frac{n}{2}}K^3. \quad (A.7)$$

### A.4 Calculation of the trace with spin-field interactions

Now we turn to the expression (5.18) within an elementary triangle, which is comparable to (4.18), but with spin-field interactions included. We have to take the trace over this expression over the inner spins $\sigma_4$, $\sigma_5$ and $\sigma_6$. This will yield a similar expression for the spins of the lower generations, from which the renormalisation equations can be derived.

As explained on page 131 we only need to retain in this trace the terms
which are linear in $H$, $M$, $R$ and $S$. Further we can put $L = 0$ right from the start. Thus we only calculate the spin-field interaction terms which are linear in $H$, $M$, $R$ and $S$, and retain also only the lowest order contributions in $q$ for each kind of interaction term.

Again we will use a graphical representation. The expansion of (5.18) yields $2^{39}$ terms. These contain all possible combinations of the interactions that appear in (5.18). Hence we can again represent each term by a graph and consider all possible graphs on an elementary triangle containing (or not containing) any possible number of:

- simple bonds, each of which contributes a weight factor $q^{\frac{1}{2}}K$,

- sites that are fixed to $\sigma = 1$. Here we have to be careful and distinguish between the inner sites and corner sites of the elementary triangle:
  
  - inner sites with $\sigma = 1$: these contribute a weight factor $q^aH$,
  
  - corner sites with $\sigma = 1$: the contribution they give should be divided over the two elementary triangles that any site is part of (except for sites in the highest and lowest generations), hence such sites contribute a weight factor $q^aH/2$

- bonds that connect two sites that are fixed to $\sigma = 1$, each such a bond contributes weight factor $q^\delta M$,

- upright triangular clusters in which one ordinary bond appears, and the remaining site is fixed to $\sigma = 1$, each such a triangle contributes a weight factor $q^\gamma R$,

- upright triangular clusters in which all sites are fixed to $\sigma = 1$, each such a triangle contributes a weight factor $q^\beta S$.

Again performing the trace adds a factor $q$ per free spin or per free cluster of spins. Spins that take on $\sigma = 1$ are of course not free.

Of course writing out all possible graphs is not really sensible. By making the right considerations, one can see already soon which kind of graphs come into consideration to generate each kind of interaction term. Since we can restrict ourselves to linear expressions in $H$, $M$, $R$ and $S$, only one of the above listed interactions can appear at once in a graph, except for the simple bonds. What lasts then is to fix the number of bonds that makes each graph of lowest order in $q$ (considering the weight per bond and the eventual free spins in the graph).
A.4. THE TRACE WITH SPIN-FIELD INTERACTIONS

In tables A.4-A.7 we list per generated spin-field interaction term the ingredients (spin-field interactions plus number of bonds) in the graphs that produce the lowest order contribution to this kind of interaction. Each column stands for the single spin-field interaction included, and in the first row we list the number of bonds in these graphs. The second row lists the weight factor of these graphs, and the third row the actual number of graphs within each category that generate a lowest order term. Again we mention some 'good' and 'bad' examples of graphs. By bad examples we mean graphs that have the right ingredients but whose configuration does not make them of lowest order in $q$.

What concerns the notation in the tables we note that

- when the interaction that fixes a spin to $\sigma = 1$ is included, an explicit sum is given. In this sum the first number is the number of graphs in which an inner spin is fixed to $\sigma = 1$, and the denominator of the second number is the number of graphs in which an corner spin is fixed to $\sigma = 1$. This last number has to be divided by 2 since the weight factor of such graphs is proportional to $H/2$;

- $b_i = 1$ stands for bond $i$ in which the two spins are fixed to $\sigma = 1$. 
<table>
<thead>
<tr>
<th>Included interaction:</th>
<th>$\delta_{\sigma_i,1}$</th>
<th>$\delta_{\sigma_i\sigma_j,1}$</th>
<th>$\delta_{\sigma_i\sigma_j,\sigma_k1}$</th>
<th>$\delta_{\Delta,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of bonds</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>weight</td>
<td>$q^{\alpha+\frac{3}{2}}K^3H$</td>
<td>$q^{\beta+1}K^2M$</td>
<td>$q^{\gamma+1}K^2R$</td>
<td>$q^{\delta+\frac{1}{2}}KS$</td>
</tr>
<tr>
<td>number of graphs</td>
<td>$50 + \frac{50}{2}$</td>
<td>50</td>
<td>50</td>
<td>4</td>
</tr>
<tr>
<td>good examples</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_4 = 1$</td>
<td>$b_1 = 1$</td>
<td>$\sigma_1 = 1 + b_3$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+b_1b_2b_6$</td>
<td>$+b_2b_5$</td>
<td>$+b_1b_5$</td>
<td>$+b_5$</td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_4 = 1$</td>
<td>$b_3 = 1$</td>
<td>$\sigma_4 = 1 + b_2$</td>
<td>$\Delta_2 = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+b_1b_2b_6$</td>
<td>$+b_1b_5$</td>
<td>$+b_1b_5$</td>
<td>$+b_7$</td>
</tr>
<tr>
<td>bad examples</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_4 = 1$</td>
<td>$b_1 = 1$</td>
<td>$\sigma_1 = 1 + b_3$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+b_1b_3b_7$</td>
<td>$+b_1b_5$</td>
<td>$+b_1b_4$</td>
<td>$+b_4$</td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_4 = 1$</td>
<td>$b_3 = 1$</td>
<td>$\sigma_4 = 1 + b_3$</td>
<td>$\Delta_2 = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+b_1b_8b_9$</td>
<td>$+b_4b_5$</td>
<td>$+b_5b_7$</td>
<td>$+b_8$</td>
</tr>
</tbody>
</table>

Table A.4: Lowest order contributions to the generated $\delta_{\sigma_i,1}$ type of interaction after tracing over (5.18).
A.4. THE TRACE WITH SPIN-FIELD INTERACTIONS

<table>
<thead>
<tr>
<th>Included interaction:</th>
<th>$\delta_{\sigma_i,1}$</th>
<th>$\delta_{\sigma_i\sigma_j,1}$</th>
<th>$\delta_{\sigma_i\sigma_j,\sigma_h1}$</th>
<th>$\delta_{\Delta,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of bonds</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>weight</td>
<td>$q^\alpha+2K^4H$</td>
<td>$q^{\alpha+\frac{3}{2}}K^3M$</td>
<td>$q^{\gamma+\frac{3}{2}}K^3R$</td>
<td>$q^{\delta+1}K^2S$</td>
</tr>
<tr>
<td>number of graphs</td>
<td>$78 + \frac{60}{2}$</td>
<td>108</td>
<td>90</td>
<td>12</td>
</tr>
<tr>
<td>good examples</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>basic interaction</td>
<td>$\sigma_4 = 1$</td>
<td>$b_1 = 1$</td>
<td>$\sigma_1 = 1 + b_3$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds</td>
<td>$+b_1b_2b_4b_5$</td>
<td>$+b_2b_4b_5$</td>
<td>$+b_1b_4b_5$</td>
<td>$+b_4b_5$</td>
</tr>
<tr>
<td>basic interaction</td>
<td>$\sigma_1 = 1$</td>
<td>$b_3 = 1$</td>
<td>$\sigma_6 = 1 + b_1$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds</td>
<td>$+b_1b_2b_4b_5$</td>
<td>$+b_1b_4b_5$</td>
<td>$+b_2b_4b_5$</td>
<td>$+b_6b_8$</td>
</tr>
<tr>
<td>bad examples</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>basic interaction</td>
<td>$\sigma_1 = 1$</td>
<td>$b_1 = 1$</td>
<td>$\sigma_1 = 1 + b_3$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds</td>
<td>$+b_1b_2b_4b_7$</td>
<td>$+b_2b_3b_4$</td>
<td>$+b_1b_2b_4$</td>
<td>$+b_5b_6$</td>
</tr>
<tr>
<td>basic interaction</td>
<td>$\sigma_4 = 1$</td>
<td>$b_3 = 1$</td>
<td>$\sigma_6 = 1 + b_1$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds</td>
<td>$+b_1b_2b_3b_4$</td>
<td>$+b_1b_5b_6$</td>
<td>$+b_4b_5b_8$</td>
<td>$+b_6b_9$</td>
</tr>
</tbody>
</table>

Table A.5: Lowest order contributions to the generated $\delta_{\sigma_i\sigma_j,1}$ type of interaction after tracing over (5.18).
<table>
<thead>
<tr>
<th>Included interaction:</th>
<th>$\delta_{\sigma_{1},1}$</th>
<th>$\delta_{\sigma_{i},\sigma_{j},1}$</th>
<th>$\delta_{\sigma_{i},\sigma_{j},\sigma_{k}1}$</th>
<th>$\delta_{\Delta,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of bonds</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>weight</td>
<td>$q^{\alpha+2}K^{4}H$</td>
<td>$q^{\alpha+\frac{3}{2}}K^{3}M$</td>
<td>$q^{\gamma+\frac{3}{2}}K^{3}R$</td>
<td>$q^{\delta+1}K^{2}S$</td>
</tr>
<tr>
<td>number of graphs</td>
<td>$12 + \frac{30}{2}$</td>
<td>12</td>
<td>30</td>
<td>1</td>
</tr>
<tr>
<td>good examples</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_{1} = 1$</td>
<td>$b_{3} = 1$</td>
<td>$\sigma_{1} = 1 + b_{3}$</td>
<td>$\Delta_{1} = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+b_{1}b_{2}b_{8}b_{9}$</td>
<td>$+b_{2}b_{8}b_{9}$</td>
<td>$+b_{5}b_{7}b_{8}$</td>
<td>$+b_{8}b_{9}$</td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_{4} = 1$</td>
<td>$b_{3} = 1$</td>
<td>$\sigma_{4} = 1 + b_{2}$</td>
<td>$\Delta_{2} = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
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<td>$+b_{1}b_{8}b_{9}$</td>
<td>$+b_{4}b_{6}b_{7}$</td>
<td>$+b_{2}b_{7}$</td>
</tr>
<tr>
<td>bad examples</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_{1} = 1$</td>
<td>$b_{1} = 1$</td>
<td>$\sigma_{1} = 1 + b_{3}$</td>
<td>$\Delta_{1} = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+b_{1}b_{4}b_{6}b_{8}$</td>
<td>$+b_{2}b_{7}b_{8}$</td>
<td>$+b_{1}b_{4}b_{5}$</td>
<td>$+b_{2}b_{6}$</td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_{4} = 1$</td>
<td>$b_{3} = 1$</td>
<td>$\sigma_{4} = 1 + b_{2}$</td>
<td>$\Delta_{1} = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+b_{1}b_{4}b_{6}b_{7}$</td>
<td>$+b_{1}b_{4}b_{5}$</td>
<td>$+b_{1}b_{6}b_{9}$</td>
<td>$+b_{4}b_{5}$</td>
</tr>
</tbody>
</table>

Table A.6: Lowest order contributions to the generated $\delta_{\sigma_{i},\sigma_{j},\sigma_{k}1}$ type of interaction after tracing over (5.18).
### Included interaction:

<table>
<thead>
<tr>
<th></th>
<th>$\delta_{\sigma_i,1}$</th>
<th>$\delta_{\sigma_i\sigma_j,1}$</th>
<th>$\delta_{\sigma_i\sigma_j\sigma_k1}$</th>
<th>$\delta_{\Delta,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of bonds</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>weight</td>
<td>$q^{\alpha+\frac{5}{2}}K^5H$</td>
<td>$q^{\beta+2}K^4M$</td>
<td>$q^{\gamma+2}K^4R$</td>
<td>$q^{\delta+\frac{3}{2}}K^3S$</td>
</tr>
<tr>
<td>number of graphs</td>
<td>$162 + \frac{162}{2}$</td>
<td>270</td>
<td>270</td>
<td>36</td>
</tr>
<tr>
<td>good examples</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_1 = 1$</td>
<td>$b_1 = 1$</td>
<td>$\sigma_1 = 1 + b_3$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+ b_1 b_2 b_4 b_5 b_7$</td>
<td>$+ b_2 b_4 b_5 b_7$</td>
<td>$+ b_1 b_4 b_8 b_9$</td>
<td>$+ b_4 b_5 b_7$</td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_2 = 1$</td>
<td>$b_3 = 1$</td>
<td>$\sigma_4 = 1 + b_2$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+ b_2 b_4 b_7 b_8 b_9$</td>
<td>$+ b_1 b_4 b_8 b_9$</td>
<td>$+ b_4 b_7 b_8 b_9$</td>
<td>$+ b_4 b_7 b_9$</td>
</tr>
<tr>
<td>bad examples</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_1 = 1$</td>
<td>$b_1 = 1$</td>
<td>$\sigma_1 = 1 + b_3$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+ b_1 b_2 b_3 b_4 b_5$</td>
<td>$+ b_2 b_3 b_4 b_5$</td>
<td>$+ b_1 b_2 b_4 b_5$</td>
<td>$+ b_3 b_5 b_6$</td>
</tr>
<tr>
<td>basic interaction:</td>
<td>$\sigma_1 = 1$</td>
<td>$b_3 = 1$</td>
<td>$\sigma_4 = 1 + b_2$</td>
<td>$\Delta_1 = 1$</td>
</tr>
<tr>
<td>additional bonds:</td>
<td>$+ b_1 b_2 b_4 b_6 b_8$</td>
<td>$+ b_4 b_7 b_8 b_9$</td>
<td>$+ b_1 b_3 b_4 b_5$</td>
<td>$+ b_4 b_6 b_8$</td>
</tr>
</tbody>
</table>

Table A.7: Lowest order contributions to the generated $\delta_{\Delta,1}$ type of interaction after tracing over (5.18).
Summary

The concept of self-organised criticality (SOC), introduced by Bak, Tang and Wiesenfeld [17] in 1987, is an attempt to describe self-organisation in complex systems. It intends to describe a diversity of natural systems that exhibit scaling behaviour, similar to that of equilibrium systems at criticality. However these systems are far from equilibrium and in constant interaction with their environment.

To illustrate and investigate the theory toy models were introduced that were presumed to exhibit SOC. Sandpile models are among the most important ones of these models. Sandpiles are cellular automata defined on a lattice. They are investigated by analytical methods and computer simulation. Their dynamics is highly nonlinear and is governed by threshold mechanisms and avalanches. Thus far, the most developed analytical theory on sandpiles is the group theoretical one of the Bak-Tang-Wiesenfeld (BTW) sandpile model by Deepak Dhar [27], which is able to give a good characterisation of the SOC state. It also lays a connection between the BTW model and the Potts model, a magnetic model from equilibrium statistical mechanics.

Sandpile models have been extensively studied on the two dimensional square lattice. Most interestingly, De Menech and Stella [33, 35] found that the BTW model does not exhibit finite-size scaling, but instead obeys a more complicated - multifractal - scaling. On the other hand, the group of Priezzhev introduced the concept of waves of topplings [39]. These are basic constituents of avalanches that are remarkably simple to analyse. They exhibit finite-size scaling on the twodimensional square lattice. Waves are strongly correlated and somehow these correlations are responsible for the complex, multiscaling behaviour of avalanches. Theoretical arguments regarding the scaling behaviour of the lattice Green function indicate that the correlations between waves are only significant on lattices with $D_{rw} \geq D_f$, i.e. where the exponent describing the spatio-temporal scaling of a random walker is larger than, or equals, the fractal dimension of the lattice. This is why avalanches become simpler on Euclidian lattices with $d > 2$ [46].
With this in mind it is not hard to find motivations to study sandpile models on fractal lattices. To begin with, fractals are omnipresent in Nature, therefore it is important to study dynamical processes on fractal supports. Fractal lattices provide an occasion to verify conjectures made from the study of sandpiles on the two dimensional square lattice. Most interesting, there exist fractal lattices with $D_{rw} > D_f$, where the correlations between waves are more pronounced as in the two dimensional case, for example the Sierpinski gasket.

It would be interesting to find a dynamical renormalisation procedure in order to describe the complex dynamics of the sandpile models. Fractal lattices are very suitable for real space renormalisation. It is in fact possible to perform exact real space renormalisation calculations of the Potts model on various fractal lattices, from which various steady state BTW properties can be calculated exactly.

In this thesis first the existing sandpile theory is summarised, and extended to the case of a general fractal lattice (chapter 3). Doing this, some general results are derived for the BTW model. These include the asymptotic behaviour of the distribution function of waves and of the last wave in an avalanche, a scaling relation for the expected number of topplings and of the number of waves in an avalanche, and a detailed description of the conditional probability functions of successive waves.

In a case study of the BTW model on the Sierpinski gasket the number of sandpile configurations that appear in the SOC state is calculated exactly by a renormalisation calculation of the Potts model. Using the exact lattice Green function the probability for any site in the lattice to be occupied by 1 grain of sand is calculated for Sierpinski gaskets of finite order. Furthermore exact results for the expected number of topplings and of the number of waves in an avalanche were derived.

After these theoretical considerations results from numerical simulations are presented. It is found that distribution functions on a fractal lattice exhibit various irregularities which makes it difficult to make collapse plots. Such plots are traditionally used to examine scaling behaviour in the models. Here the methods of moment scaling and multifractal analysis are applied instead and found very appropriate to analyse the scaling behaviour, as well as to determine exponents in the case of finite-size scaling. These methods allow to conclude that on the Sierpinski gasket the wave and last wave distributions in the BTW model follow finite-size scaling. To the contrary, avalanches in this model are found not to obey finite-size scaling, but multifractal scaling instead, as is the case on the two dimensional square lattice.

The scaling forms for successive wave conditional probability functions
conjectured in [49, 50] are found likely to hold, as well as the relations between exponents in these scaling forms derived in chapter 3. Strong long-range correlations between waves in the BTW model are detected.

Finally, avalanches in the Manna model are found to follow finite-size scaling, classifying this model in another universality class than the BTW model, as was already found in two dimensions [33, 35].

As a last part of this thesis a study of the BTW model with dissipation is presented. By mapping this model on a random walk problem, this model is proven to be not critical for any amount of dissipation rate. The exponent describing the crossover to criticality when the dissipation rate vanishes is found to be $1/D_{rw}$. In the context of this chapter, the magnetic exponent of the Potts model on a Sierpinski gasket is calculated with a renormalisation method.
Samenvatting


Om de theorie te illustreren en verder te onderzoeken werden modellen ingevoerd waarvan verondersteld werd dat ze SOC vertonen. Een belangrijke categorie van deze modellen vormen de zandhoopmodellen. Zandhopen zijn cellulaire automaten, gedefinieerd op een rooster. Ze worden bestudeerd met analytische methodes en computersimulaties. Hun dynamica is sterk niet-linear en wordt bepaald door drempelmechanismen en lawines. De tot nog toe meest succesvolle theorie over zandhopen is de groepetheoretische benadering voor het Bak-Tang-Wiesenfeld (BTW) model van Deepak Dhar [27]. Deze theorie is in staat een goede karakterisatie van de SOC toestand te geven. Ze legt ook een verband tussen het BTW model en het Potts model, een magnetisch model uit de evenwichts statistische mechanica.

Zandhoopmodellen zijn tot nog toe uitgebreid onderzocht op het tweedimensionaal vierkant rooster. Erg interessant is de vondst van De Menech en Stella [33, 35] dat het BTW model geen 'finite size' schaling volgt, maar een meer complexe vorm van schaling, namelijk multifractale schaling. Daarnaast introduceerde de groep rond Priezzhev eerder al het concept van 'golven van tuimelingen' [39]. Deze golven zijn basisbestanddelen van lawines, die opmerkelijk eenvoudig te analyseren zijn. Ze volgen een 'finite size' schaling op het tweedimensionaal vierkant rooster. Golven zijn sterk gecorreleerd in de tijd en op een of andere manier zijn deze correlaties verantwoordelijk voor het complexe schalingsgedrag van lawines. Theoretische argumenten betreffende het schalingsgedrag van de Greense functie wijzen erop dat de correlaties tussen golven enkel significant zijn op roosters waar $D_{rw} \geq D_f$, dus waar de exponent die de tijd-afstand schaling van een random wandelaar
beschrijft groter of gelijk is aan de fractale dimensie van het rooster. Daarom ook worden lawines eenvoudiger op Euclidische roosters met $d > 2$ [46].

Dit wetende, is het niet zo moeilijk om motivaties te vinden voor een studie van zandhopen op fractale roosters. Vooreerst zijn fractalen alomtegenwoordig in de Natuur, en daarom is het uiteraard belangrijk om dynamische processen te bestuderen op fractale structuren. Fractalen bieden daarbij ook de mogelijkheid om vermoedens te verifiëren die gedaan werden vanuit de studie van zandhopen op het tweedimensionaal vierkant rooster. Er bestaan trouwens fractale roosters waarop $D_{rw} > D_f$, dus waarop de correlaties tussen golven meer uitgesproken zijn dan in het tweedimensionaal geval. Een voorbeeld is de Sierpinski zeef.

Het zou erg interessant zijn moest er een dynamische renormalisatieprocedure gevonden kunnen worden om de complexe zandhoopdynamica te kunnen bestuderen. Ook hierin ligt het belang van fractalen, dat zij erg geschikt zijn voor positie-ruimte renormalisatie berekeningen. Het is bijvoorbeeld mogelijk om dergelijke renormalisatie berekeningen exact uit te voeren voor het Potts model op verscheidene fractale roosters, waaruit bepaalde eigenschappen van het BTW model exact kunnen afgeleid worden.

In deze thesis wordt eerst de bestaande zandhooptheorie kort herhaald, en waar nodig uitgebreid naar de situatie van een algemeen fractaal rooster (hoofdstuk 3). Bij deze aanpassing worden er een aantal algemene resultaten voor het BTW model afgeleid. Deze omvatten een beschrijving van het asymptomatische gedrag van de verdelingsfuncties voor golven en voor de laatste golf binnen een lawine, een schalingsrelatie voor het gemiddeld aantal tuimelingen en het gemiddeld aantal golven in een lawine, en een gedetailleerde beschrijving van de statistische correlaties van opeenvolgende golven.

In een specifieke studie van het BTW model op de Sierpinski zeef wordt het aantal configuraties dat in de SOC toestand optreedt exact berekend via renormalisatie van het Potts model. Gebruik makende van de Greense functie wordt tevens de exacte waarschijnlijkheid berekend dat eender welk roosterpunt door precies 1 zandkorreltje wordt bezet, en dit voor het geval van Sierpinski zeven van een eindige orde. Ook het gemiddeld aantal tuimelingen en het gemiddeld aantal golven in een lawine worden exact berekend.

Na deze theoretische beschouwingen worden de resultaten van numerieke simulaties voorgesteld. Een belangrijke vaststelling is dat bepaalde verdelingsfuncties op fractale roosters nogal wat onregelmatigheden vertonen. Deze maken klassieke methodes om bijvoorbeeld schalingsgedrag te analyseren moeilijker toepasbaar. Daarom worden hier de methodes van schaling van momenten en multifractale analyse toegepast. Deze blijken erg succesvol om bijvoorbeeld schalingsgedrag vast te stellen of om exponenten te bepalen.
in geval van 'finite-size' schaling. De resultaten van deze analyses laten toe
te concluderen dat op de Sierpinski zeef de verdelingsfuncties van golven en
laatste golven in het BTW model 'finite size' schaling volgen. Daarentegen
doen de lawines in dit model dit duidelijk niet. Ze vertonen een multifractaal
schalingsgedrag zoals op het tweedimensionaal vierkant rooster.

De schalingsvormen voor de correlaties tussen opeenvolgende golven zoals
deze voorgesteld werden in [49, 50] lijken ook op de Sierpinski zeef van
toepassing. De relaties tussen exponenten in deze schalingsvormen die in
hoofdstuk 3 afgeleid werden lijken voldaan. Er worden tevens sterke lange-
tijdscorrelaties vastgesteld tussen golven in het BTW model.

Ten slotte geven deze analyses duidelijk aan dat de lawines in het Manna
model 'finite size' schaling volgen. Dit model hoort dus in een andere uni-
versaliteitsklasse thuis dan het BTW model, zoals reeds eerder in twee dimensies
werd vastgesteld [33, 35].

Een laatste deel van deze thesis omvat de studie van het BTW model met
dissipatie. Door het model af te beelden op het probleem van een random
wandelaar wordt duidelijk aangetoond dat het model niet kritisch kan zijn
voor eender welke mate van dissipatie. De exponent die de overgang naar het
kritisch model beschrijft wordt theoretisch bepaald op $1/D_{ru}$. Dit wordt aan
de hand van exacte gegevens bevestigd op de Sierpinski zeef. In de context
van deze studie wordt tevens de magnetische exponent van het Potts model
op een Sierpinski zeef berekend met een renormalisatie methode.
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